Sequentialization for Multiplicative Proof-Nets

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(developped with O. Laurent, L. Tortora de Falco, L. Vaux Auclair)

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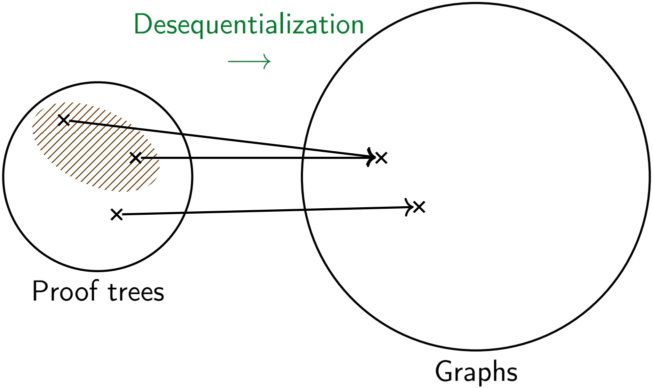
14 October 2025

Introduction

Proof nets: graphical syntax for proofs of **Linear Logic**, *canonical*



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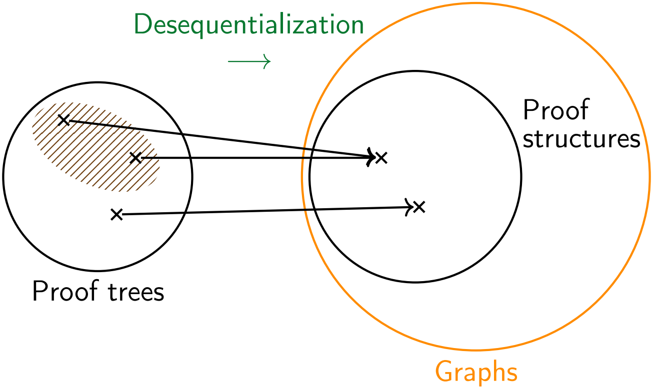

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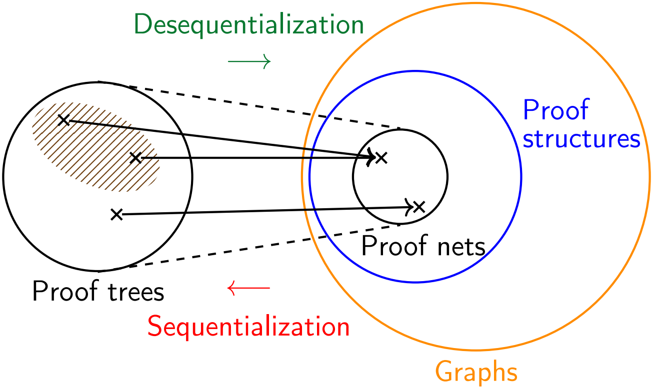

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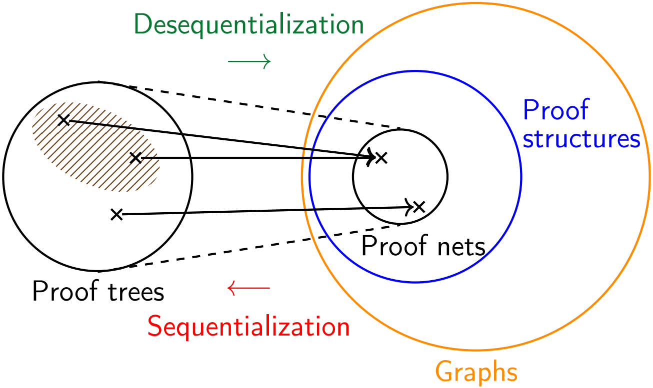

Multiple **correctness criteria**, proofs of sequentialization

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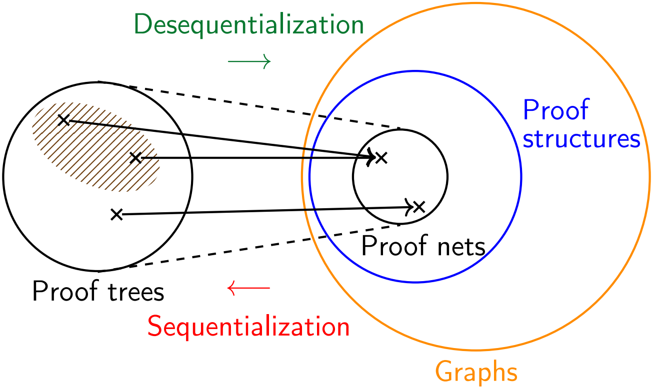

Multiple **correctness criteria**, proofs of sequentialization

Today: an "easy" proof of sequentialization

Outline

- Quick reminder: Proof-Nets & Desequentialization

- Proof of Sequentialization

Unit-free Multiplicative Linear Logic

Formulas

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A$$

Orthogonal

$$(X^{\perp})^{\perp} = X$$
$$(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$
$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

Rules

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr)$$

Example of proof structure by desequentialization

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash Y, Y^\perp$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash X \otimes Y, X^\perp, Y^\perp$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash Z^\perp, Z$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash X \otimes Y, X^\perp, Y^\perp \otimes Z^\perp, Z$}
\RightLabel{$(\parr)$}
\UnaryInfC{$\vdash X \otimes Y, X^\perp \parr (Y^\perp \otimes Z^\perp), Z$}
\RightLabel{$(\parr)$}
\UnaryInfC{$\vdash X \otimes Y, (X^\perp \parr (Y^\perp \otimes Z^\perp)) \parr Z$}
\end{prooftree}
$$

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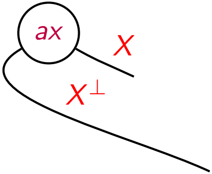
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}\,(ax) \qquad \dfrac{}{\vdash Y, Y^\perp}\,(ax)}{\vdash X \otimes Y, X^\perp, Y^\perp}\,(\otimes) \qquad \dfrac{}{\vdash Z^\perp, Z}\,(ax)}{\vdash X \otimes Y, X^\perp, Y^\perp \otimes Z^\perp, Z}\,(\otimes)}{\vdash X \otimes Y, X^\perp \parr (Y^\perp \otimes Z^\perp), Z}\,(\parr)}{\vdash X \otimes Y, (X^\perp \parr (Y^\perp \otimes Z^\perp)) \parr Z}\,(\parr)$$



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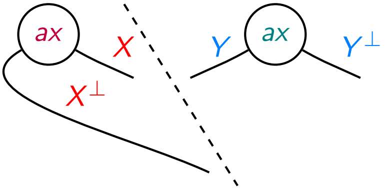

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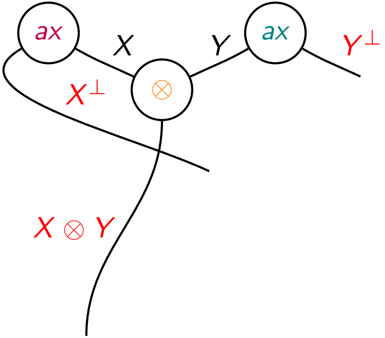

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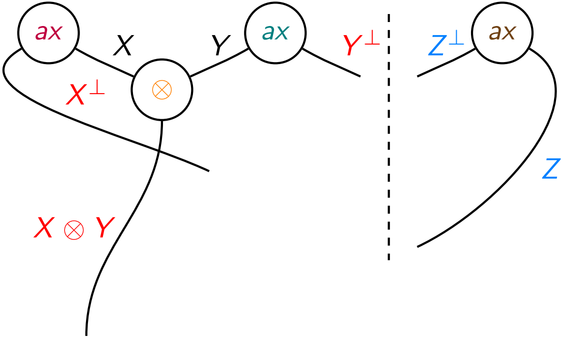

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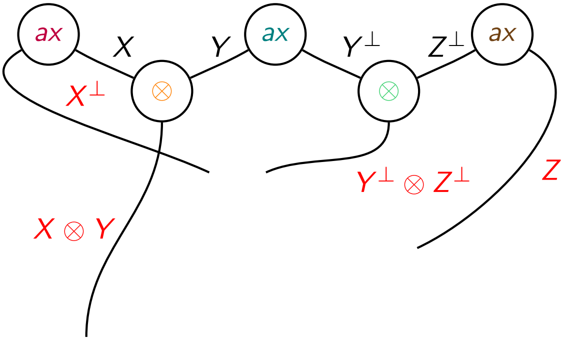

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$$
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    \dfrac{
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        \dfrac{}{\vdash X^{\perp}, X}\,(ax)
        \qquad
        \dfrac{}{\vdash Y, Y^{\perp}}\,(ax)
      }{\vdash X \otimes Y, X^{\perp}, Y^{\perp}}\,(\otimes)
      \qquad
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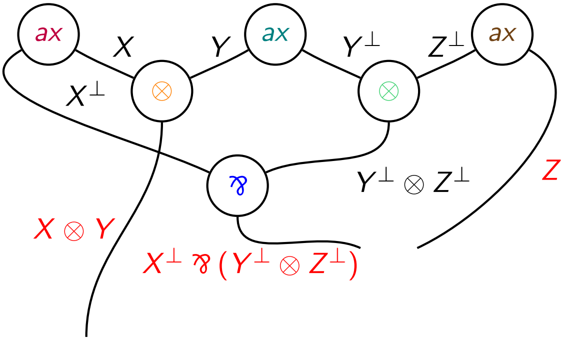

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      \qquad
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    }{\vdash X \otimes Y, X^\perp, Y^\perp}\,(\otimes)
    \qquad
    \dfrac{}{\vdash Z^\perp, Z}\,(ax)
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}{
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}\,(\parr)
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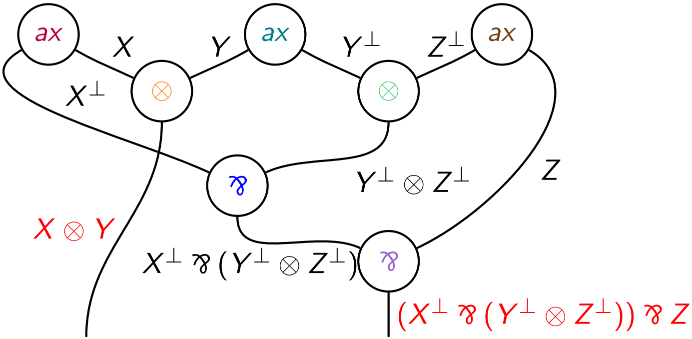
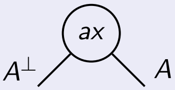


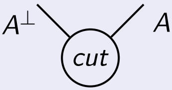
Proof structure

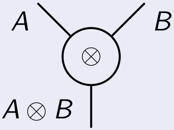
Definition

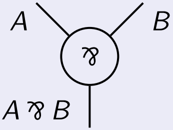
Partial graph with labels on vertices $\rightarrow$ *ax* / *cut* / $\otimes$ / $\parr$

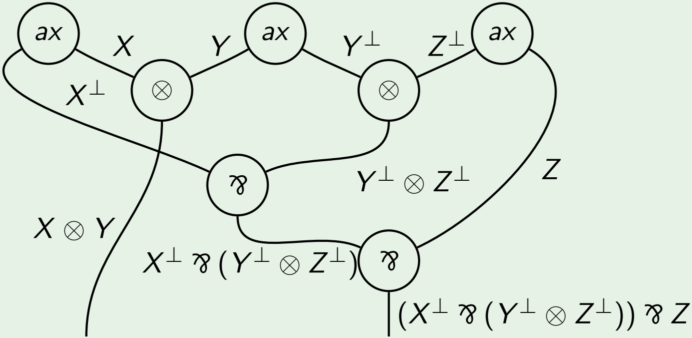
on edges $\rightarrow$ formula



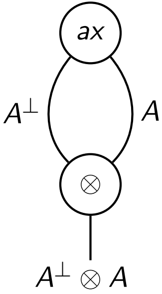


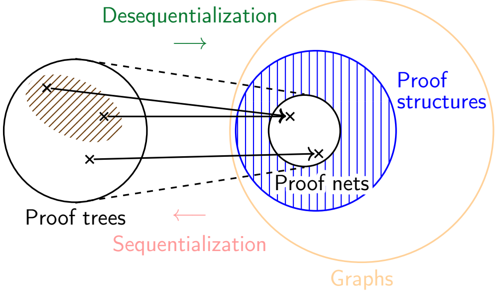






Correctness





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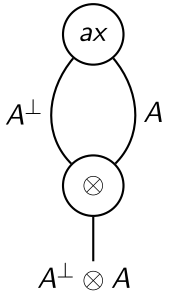
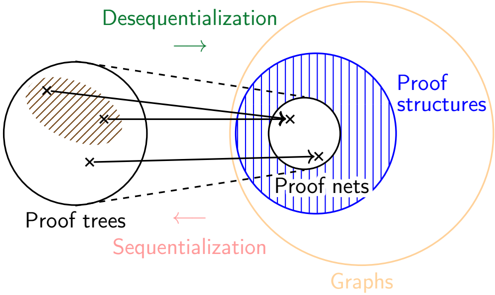
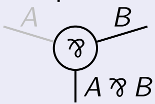




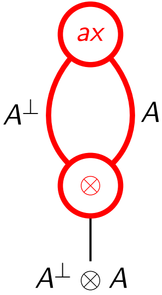
Danos-Regnier Correctness Criterion

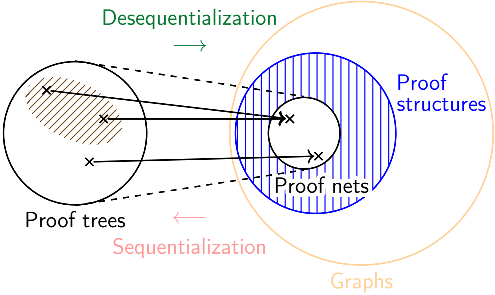
Correctness graph: remove one premise of each ⅋



Correct if all correctness graphs are *acyclic* and *connected*

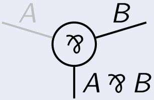
Correctness





Danos-Regnier Correctness Criterion

Correctness graph: remove one premise of each $⅋$



Correct if all correctness graphs are *acyclic* and *connected*

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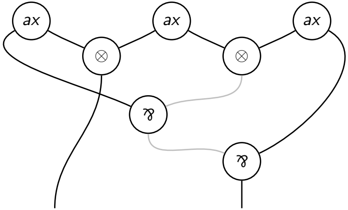

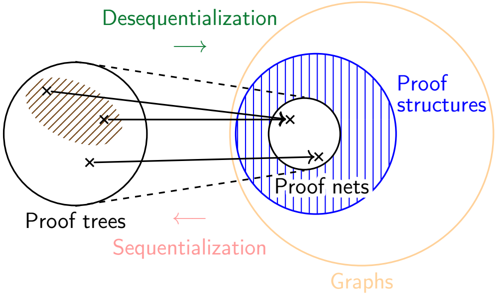

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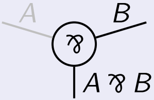
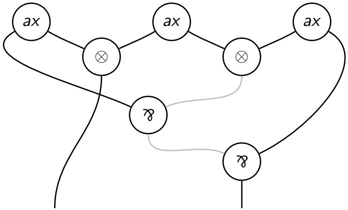
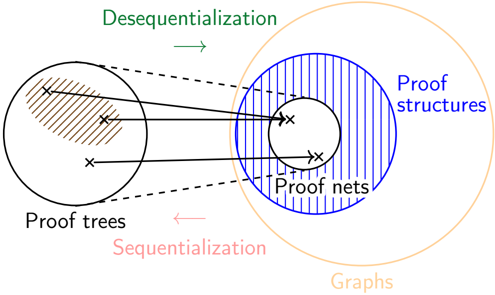


Correct if all correctness graphs are *acyclic* and *connected*

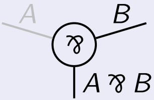
Correctness





Danos-Regnier Correctness Criterion

Correctness graph: remove one premise of each $\parr$



Correct if all correctness graphs are *acyclic* and *connected*

Switching path: does not contain the two premises of a $\parr$

$\rightsquigarrow$ No switching cycle

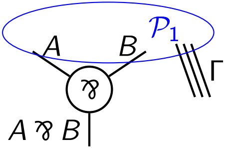
Sequentialization

Theorem: Sequentialization

Given a correct proof structure, there is a proof desequentializing to it.

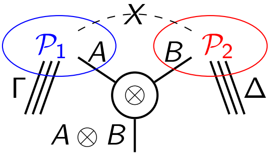
How to prove it? Find a vertex corresponding to a **last rule**

- a $\parr$ with no vertex below



$$\rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\ (\parr)$$

- a $\otimes$ with no vertex below and not in a cycle



$$\rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash A, \Gamma} \qquad \overset{\pi_2}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes)$$

- a *cut* not in a cycle
  $\rightsquigarrow$ same as a $\otimes$, we will not consider them

Building a Switching Path

Idea of the proof: build a path p by following a chain of **dependencies**
$\rightsquigarrow$ before doing this $\otimes$-rule, must first do this $\parr$-rule, before which we must do this $\otimes$-rule ... stopping on a rule we can actually do *(previous drawings)*

Building a Switching Path

Idea of the proof: build a path p by following a chain of **dependencies** $\rightsquigarrow$ before doing this $\otimes$-rule, must first do this $⅋$-rule, before which we must do this $\otimes$-rule ... stopping on a rule we can actually do *(previous drawings)*

Outline:

1. We build a switching path p by visiting vertices, halting our construction only on a vertex corresponding to a last rule.
2. We prove p cannot "loop" (all its vertices are different).

And we are done:

- The graph is finite $\longrightarrow$ the path's construction must stop, necessarily on a vertex corresponding to a last rule!
- We remove the found vertex and conclude by induction hypothesis.

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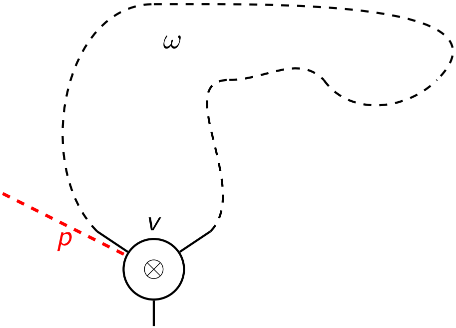
- We remove the found vertex and conclude by induction hypothesis.

Starting point for p: an arbitrary $\otimes$ or $\parr$ v with no vertex below it (if only ax, easy).

Ascending Path α: a $\otimes$ blocked by a $\parr$

Assume p ends on a vertex v with no one below it.

- *If v is a $\parr$ or a $\otimes$ not in a cycle:* we are done.
- *Otherwise:* v is a $\otimes$ in a cycle ω.
  Remark ω cannot be switching: it has at least one  ;
  w.l.o.g. ω **has a minimal number of those**.



Ascending Path α: a $\otimes$ blocked by a ⅋

Assume p ends on a vertex v with no one below it.

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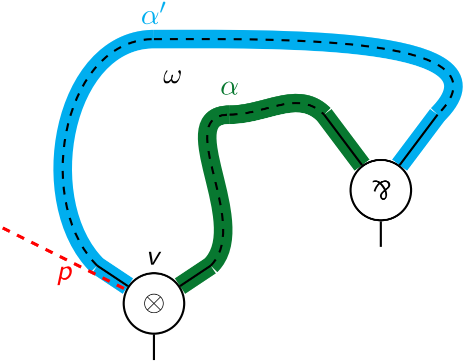
- *Otherwise:* v is a $\otimes$ in a cycle ω.
  Remark ω cannot be switching: it has at least one 

  ;
  w.l.o.g. ω **has a minimal number of those**.
  We start/extend p using the part α of ω towards the **first** 

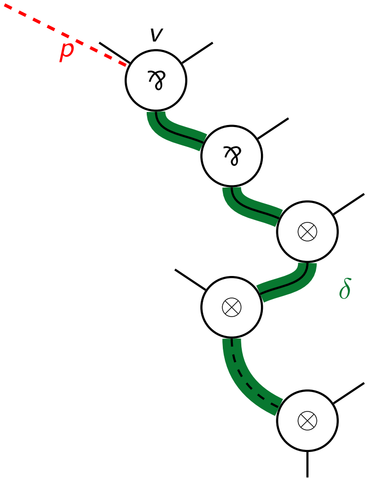
  .



Descending Path δ: formula dependency

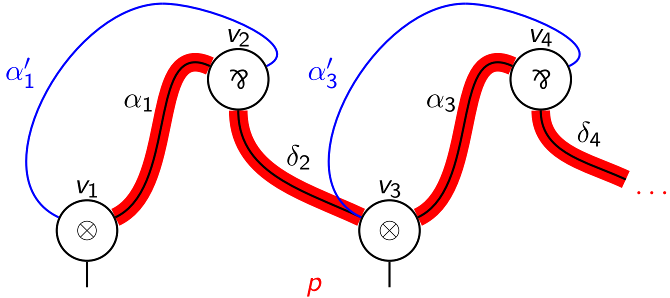
Assume p ends on a $⅋$-vertex v with someone below it.

Extend p by taking downward edges towards a vertex with no one below it.



Built Switching Path p

Concatenation of an ascending path, a descending path, ...



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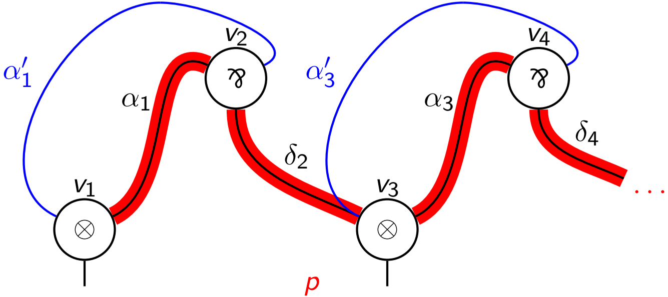

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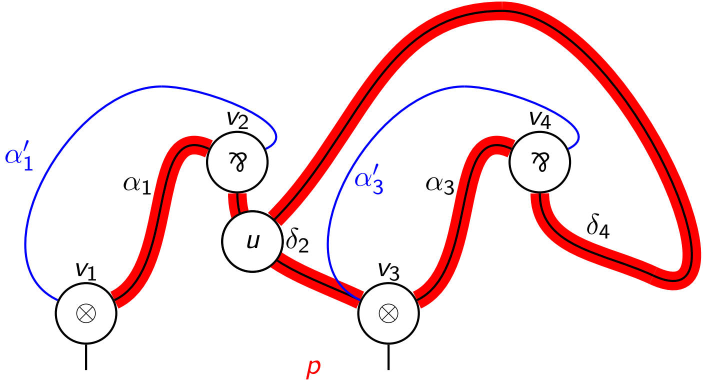
No going back on a descending path



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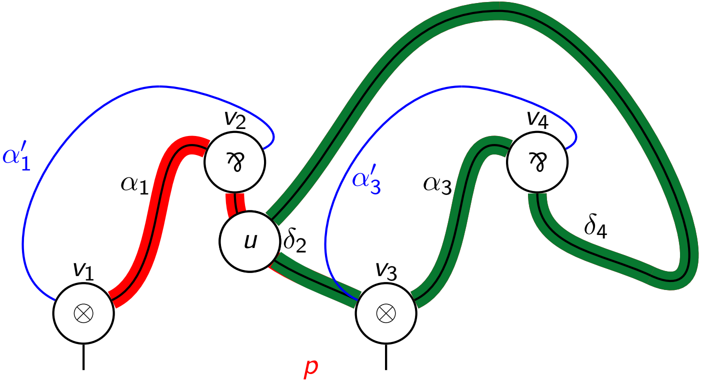
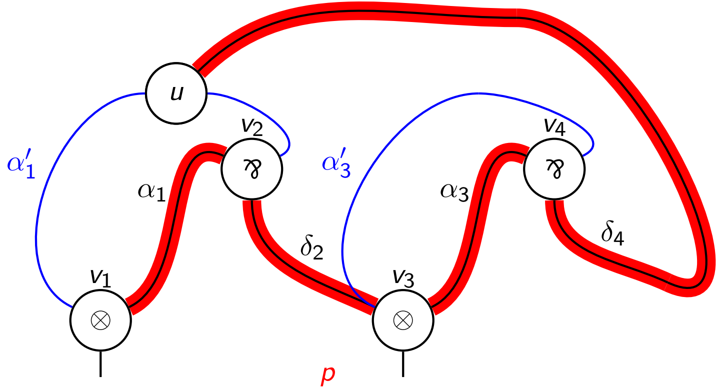


This is a switching cycle!

(No  when going back as we took a downward edge in δ_2.)

No going back on an ascending path



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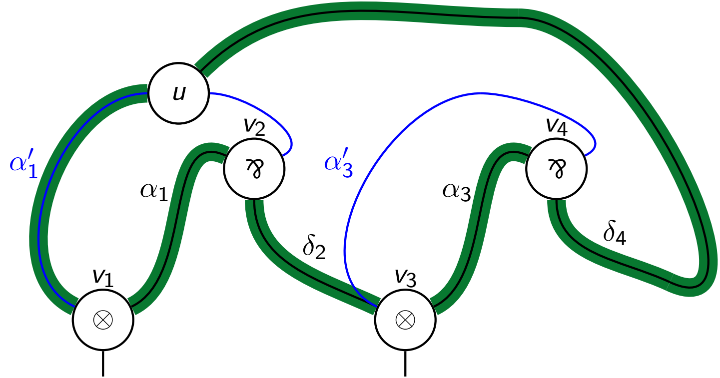
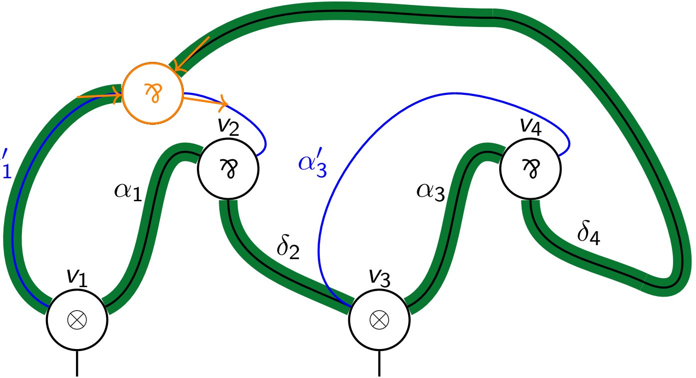


This has at least as many  **as $\alpha_1 \cdot \alpha'_1$!**

But we lost one at v_2, so:

- there is none between u and v_2; and
- the green cycle takes both premises of u.

No going back on an ascending path



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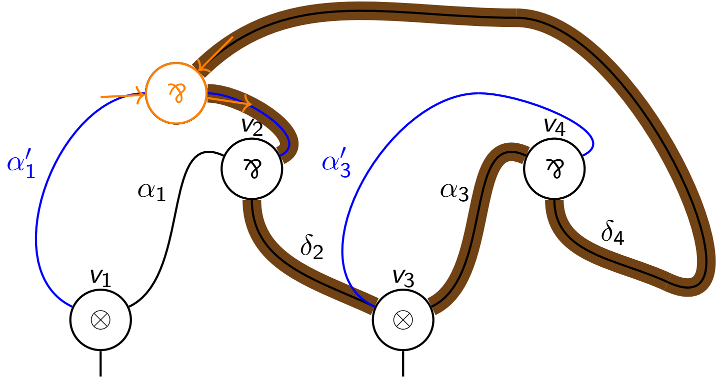
No going back on an ascending path



This has at least as many  **as $\alpha_1 \cdot \alpha_1'$!**

But we lost one at v_2, so:

- there is none between u and v_2; and
- the green cycle takes both premises of u.

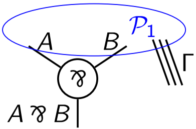
This is a switching cycle!

Summing-up

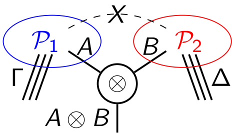
Outline:

1. We build a switching path p by visiting vertices, halting our construction only on a vertex corresponding to a last rule.
2. We prove p cannot "loop" (all its vertices are different).

$\longrightarrow$ we get a $\otimes$ or $\parr$ corresponding to a last rule, remove it and conclude by induction hypothesis.



$$\rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\ (\parr)$$



$$\rightsquigarrow \quad \dfrac{\overset{\pi_1}{\vdash A, \Gamma} \qquad \overset{\pi_2}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes)$$

- This solves the case when the proof-net has a $\parr$ or a $\otimes$.
- If it has only *ax*, easy: by correctness it has exactly one, this is the image of an *ax*-rule.

The End

- We only used no switching cycles $\longrightarrow$ this proof extends immediately to MELL proof-nets (with *mix*)!
- Proof implemented in ROCQ
- Strong links with more standard graph theory (Kotzig's theorem on perfect matchings, Yeo's theorem on edge-colored graphs, ...)