Confluence Modulo and Undecidability of Cut-Elimination in Linear Logic (Draft)

Rémi Di Guardia*

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Abstract

Proofs in a given logical system are usually considered (at least) up to cut-elimination. This is the case in linear logic, that additionally has another equivalence relation on proofs: rule commutation, which identifies proofs equal up to an irrelevant ordering of their rules. We prove these two notions coincide: two cut-free proofs are equal up to cut-elimination if and only if they are equal up to rule commutation. Our proof is modular enough to apply in many fragments of second-order linear logic, as well as in presence of axiom-expansion. Furthermore, we show that, in propositional linear logic, knowing whether two given proofs are equal up to rule commutations, or not, is undecidable—which entails that equality up to cut-elimination is undecidable.

Introduction

Linear logic, like many logics such as classical and intuitionistic logics, can be presented as a sequent calculus [Gir87] equipped with a *cut*-rule. A standard result is that the *cut*-rule is admissible, proved by presenting a rewriting procedure, called *cut-elimination*, that turns a derivation (*a.k.a.* proof tree) into another derivation with no *cut*-rule. Through the Curry-Howard correspondence, cut-elimination is associated to β-reduction in λ-calculus, hence can be seen as a form of computation. It is then sensible to consider derivations up to cut-elimination, exactly as λ-terms are often considered up to β-reduction: what matters is not really the derivation itself, but its equivalence class for equality up to cut-elimination. This is usually done in the semantics of linear logic, in particular in denotational models such as coherent spaces [Gir87] or any categorical model such as Seely categories [See89].

To consider objects up to a rewriting procedure, the royal road is to prove *strong normalization* and *confluence* of the rewriting system. The first property states that there is no infinite sequence of rewriting steps, hence any strategy for implementing the rewriting steps is guaranteed to terminate. The second property states that if different rewritings can be applied to an object o, yielding objects o_1 and o_2, then there are further rewritings of o_1 and of o_2 yielding the same object o'. These two properties are strongly linked, and strong normalization is often used to prove confluence thanks to Newman's Lemma [Ter03, Theorem 1.2.1]. Crucialy, a rewriting system with both strong normalization and confluence has the *unique normal form* property: there is a *canonical* representative of the equivalence class up to rewriting, that can be found starting from any object in the class by applying rewriting rules on it until no more step applies.

*Université Paris Cité, Inria, CNRS, IRIF, F-75013, Paris, France, remi.di.guardia@ens-lyon.org

Unfortunately, cut-elimination in linear logic is not that well-behaved: it enjoys neither strong normalization nor confluence.

No strong normalization. One of the rewriting rules defining cut-elimination is the $cut - cut$ commutative step belows, that can be repeated *ad nauseam*:

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash B^\perp, \Gamma, \Delta}\,{\scriptstyle(cut)} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,{\scriptstyle(cut)} \quad\longrightarrow\quad \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma}\,{\scriptstyle(cut)} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,{\scriptstyle(cut)}$$

No confluence. A same derivation can be rewritten through cut-elimination to two different cut-free derivations, even in unit-free multiplicative linear logic that is the simplest subsystem of linear logic:

$$\dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,{\scriptstyle(ax)} \qquad \dfrac{}{\vdash A^\perp, A}\,{\scriptstyle(ax)}}{\vdash C^\perp, C \otimes A^\perp, A}\,{\scriptstyle(\otimes)} \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,{\scriptstyle(ax)} \qquad \dfrac{}{\vdash B^\perp, B}\,{\scriptstyle(ax)}}{\vdash A^\perp, A \otimes B^\perp, B}\,{\scriptstyle(\otimes)}}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,{\scriptstyle(cut)}$$

$$\swarrow_{\times} \qquad\qquad \searrow^{\times}$$

$$\dfrac{\dfrac{}{\vdash C^\perp, C}\,{\scriptstyle(ax)} \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,{\scriptstyle(ax)} \qquad \dfrac{}{\vdash B^\perp, B}\,{\scriptstyle(ax)}}{\vdash A^\perp, A \otimes B^\perp, B}\,{\scriptstyle(\otimes)}}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,{\scriptstyle(\otimes)} \qquad \dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,{\scriptstyle(ax)} \qquad \dfrac{}{\vdash A^\perp, A}\,{\scriptstyle(ax)}}{\vdash C^\perp, C \otimes A^\perp, A}\,{\scriptstyle(\otimes)} \qquad \dfrac{}{\vdash B^\perp, B}\,{\scriptstyle(ax)}}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,{\scriptstyle(\otimes)}$$

As a consequence, the literature of linear logic about cut-elimination in sequent calculus is not as exhaustive as one may wish: there is not much more than the usual result of weak normalization, *i.e.* the data of a particular strategy that produces a cut-free proof, corresponding to the admissibility of the *cut*-rule—see for instance [Gir95; Oka99; LL22]. Nonetheless, such a result has been proved for many variations of linear logic, *c.f. e.g.* [EP16; Acc22; AMM25; BS25; BS26]. Let us mention it is possible to restrict the rewriting rules of cut-elimination so as to obtain a better-beheaved but less powerful procedure—see *e.g.* [UB99] doing so for classical logic—; we do not consider such approaches in this paper, as we want to consider cut-elimination as usually defined in linear logic. Naturally, not everything about cut-elimination revolves around normalization and confluence, and there have been many other works where cut-elimination is central, *e.g.* for complexity [MT03], around session types [DeY+12], or recently for interpolation [Sau25]. And that is not to say cut-elimination has not been studied in linear logic; on the contrary, it has been well-studied, except authors do not consider the sequent calculus presentation of linear logic, but mainly its *proof-net* [Gir96] representation. Indeed, the later is a framework solidly linked to the sequent calculus representation but where strong normalization and confluence (usually) hold, leading to a very rich literature on the subject: for instance, [Tor03; Tor01; HG05; LM08] about multiplicative-additive linear logic, [DG99; Acc13; Gue+24] about multiplicative-exponential linear logic, [Pag09; Tra09; PT17] about differential linear logic, [PT10] about linear logic with second order quantifiers, etc.

Going back to sequent calculus, since strong normalization and confluence fail we have to consider weaker properties. While strong normalization does not hold, this is only due to the $cut - cut$ rewriting step depicted beforehand. Thus, provided this step does not occur infinitely many times, one can prove strong normalization of cut-elimination in sequent calculus. Meanwhile, even if cut-elimination is not confluent, it is confluent *up to an equivalence relation.* The idea of confluence up to is fairly simple, and can even be explained using junior high school

$$
\begin{tikzcd}[column sep=tiny]
 & & & (a+b)\times(c+d) \arrow[dll] \arrow[drr] & & & \\
 & a\times(c+d)+b\times(c+d) \arrow[dl] \arrow[ddr] & & & & (a+b)\times c+(a+b)\times d \arrow[dl] \arrow[ddr] & \\
(a\times c+a\times d)+b\times(c+d) \arrow[ddr] & & & & (a\times c+b\times c)+(a+b)\times d \arrow[ddr] & & \\
 & & a\times(c+d)+(b\times c+b\times d) \arrow[dl] & & & & (a+b)\times c+(a\times d+b\times d) \arrow[dl] \\
 & (a\times c+a\times d)+(b\times c+b\times d) & & & & (a\times c+b\times c)+(a\times d+b\times d) &
\end{tikzcd}
$$

Figure 1: Rewriting graph of $(a+b)\times(c+d)$ for distributivity laws

notions and the usual exercise of expanding expressions. Consider arithmetical expressions made of variables a, b, c, etc. representing numbers, with the usual addition $+$ and multiplication $\times$. A standard exercise is the following: given an expression, expand it as much as possible using the distributivity laws $a\times(b+c)\to a\times b+a\times c$ and $(a+b)\times c\to a\times c+b\times c$. Looking at these distributivity laws as rewriting rules, one can wonder if this system is confluent. The answer is no, see for instance the rewriting graph of $(a+b)\times(c+d)$ on Figure 1. One sees there are two possible results that are completely distributed: $(a\times c+a\times d)+(b\times c+b\times d)$ and $(a\times c+b\times c)+(a\times d+b\times d)$. While these two expressions are indeed different, they are equal up to associativity and commutativity of the addition. One can prove this is a general phenomenon: while expanding an expression may not lead to a unique result, all possible results are related up to associativity and commutativity of $+$. Therefore, expanding expressions is not confluent but it is confluent modulo associativity and commutativity of $+$.

Considering cut-elimination in the sequent calculus of linear logic, we prove that, while it is not confluent, it is Church-Rosser modulo *rule commutation* (a property stronger than confluence modulo). This equivalence relation relates derivations that differ only by the order in which their rules are applied—this relates *e.g.* the two derivations from our previous example of non-confluence of cut-elimination. It is often the case that derivations are considered up to rule commutation, this relation being viewed as "bureaucracy" [Gir01] because all choices of ordering rules are adequate, and in fact are not really a choice as we wish to consider both derivations the same—but can only have two distinct derivations for it is impossible to apply two rules at once in sequent calculus. This explains why cut-elimination behaves better in proof-nets: they allow to apply two rules at the same time, and equate derivations exactly up to rule commutations [HG16]. The key result that cut-elimination is Church-Rosser modulo has so far only been proved in the restricted case of propositional multiplicative-additive linear logic [CP05, Theorem 5.1; DL25, Theorem 4.2] (with some mistakes), but was expected in the full system (*e.g.* it is claimed in [DG25, Theorem 3.8] without a proof).

This Church-Rosser modulo result leads to a simpler caracterization of equality up to cut-elimination: two cut-free derivations are equal up to cut-elimination if and only if they are related by rule commutations. Observing that our proof is (nearly) stable by sub-system, *e.g.* that proving our result for linear logic with second-order quantifiers yields also a proof for the restriction to propositional multiplicative-exponential linear logic, we chose to consider a sequent calculus of linear logic as general as possible. This means our framework is a sequent calculus with the eight connectives and units of propositional linear logic, as well as second-order quantifiers, and rules that are sometimes added to this system (*e.g.* the mix-rules). Furthermore, it is not complicated to extend our result from equality up to cut-elimination to equality up to cut-elimination and axiom-expansion.

Finaly, let us mention that equality up to rule commutation is simpler to study than equality up to cut-elimination, and its complexity has already been characterized in multiplicative linear logic [HH16] and multiplicative-additive linear logic [Bag17]. We present a very simple proof that equality up to rule commutation is *undecidable* in propositional linear logic, which entails that deciding if two derivations are equal up to cut-elimination or not is unfeasible. We do so by a reduction to provability, that has been well-studied [Lin95; Chu21; HH15] and in particular is undecidable in propositional linear logic.

Sketch of the proof. The graph on Figure 2 illustrates our principal results and their dependencies.

1. Our main contribution is that two cut-free derivations are equal up to cut-elimination if and only if they are related by rule commutation (Theorem 22).

2. As we show in a simple way that one cannot decide whether two arbitrary derivations are equal up to rule commutation (Proposition 38), our main result implies that one cannot decide whether two arbitrary derivations are equal up to cut-elimination (Proposition 39).

3. To prove our main theorem, we use two intermediate results. First, that rule commutation is included in equality up to cut-elimination (Proposition 21). Its proof is as easy as it is tedious, since the only difficulty here is the number of cases. Second, we prove cut-elimination is Church-Rosser modulo rule commutation (Proposition 20). This is the core of this paper, and it itself needs many intermediate results, that are the object of the succeeding items.

4. Our proof of Church-Rosser modulo proceeds through a lemma from rewriting theory that is not quite in the literature, but is an elementary generalization of a theorem from Aoto and Toyama [AT12] (Proposition 11). It gives us three sufficient conditions to obtain the Church-Rosser property: one about the strong normalization of some system, two about diagrams to close (that generalize local confluence and local coherence).

5. The first condition is strong normalization of cut-elimination modulo rule commutation, and more exactly of the relation "applying a cut-elimination step other than a $cut - cut$ one, then any number of $cut - cut$ steps and of *oriented* rule commutations" (Theorem 12). This part is directly built upon a work from Michele Pagani and Lorenzo Tortora de Falco [PT10] from which it is *almost* a corollary. It is quite delicate owing to the fact a natural generalization of our statement does not hold: as we will see, the relation "applying a cut-elimination step other than a $cut - cut$ one, then any number of rule commutations" is not strongly normalizing.

6. The two other conditions on closure of diagrams are big case studies, considering derivations on which are applied cut-elimination steps or rule commutations, and how to commute these steps (Lemmas 14, 15 and 18).

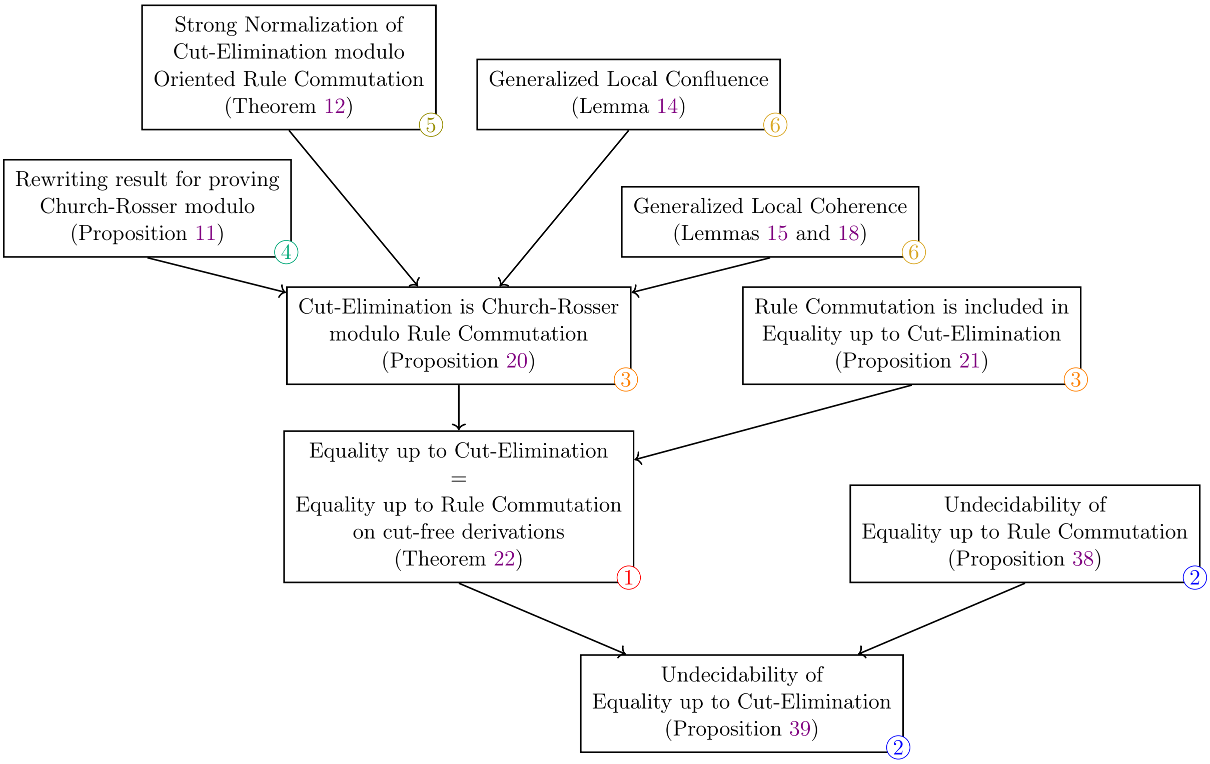


Figure 2: Principal results of this paper, along with their dependencies (with numbering associated to the sketch of the proof on Page 4)

Outline. We start by defining the sequent calculus of linear logic, with its derivations and their transformations—cut-elimination, rule commutation (Section 1). Then, we give the necessary tools from rewriting theory in presence of an equivalence relation, along with the main theorem we will use to get Church-Rosser modulo (Section 2). Afterwards, we explain how to adapt [PT10] to get the wished result on strong normalization (Section 3). Next comes the core of the paper, with the diagrams to close to conclude the proof of Church-Rosser modulo (Section 4). We then prove that rule commutation is included in equality up to cut-elimination, and deduce that equality up to cut-elimination and up to rule commutations coincide on cut-free derivations (Section 5). Our last part is about undecidability, proving that equality up to rule commutation is undecidable in propositional linear logic, and deducing that equality up to cut-elimination is also undecidable in propositional linear logic (Section 8).

1 Linear Logic: derivations and transformations of thereof

We introduce the sequent calculus syntax of linear logic [Gir87], with proofs given by derivation trees (Section 1.1). We consider a very general setting containing second order quantifiers and some additional rules:

- the mix_2- and mix_0-rules, often appearing when considering proof-nets (see *e.g.* [FR94; Ham04; Ngu20; HG05], and in particular the usual proof-nets for multiplicative-exponential linear logic [LL22]);
- the $\cup$ and $\emptyset$-rules, used in differential linear logic under the respective names $+$ and 0 [Ehr18], and the latter rule under the name ✠ in ludics [Gir01].

We study such a large logic because our main result—confluence of cut-elimination up to—is stable by sub-system, hence we can deduce confluence of many sub-systems from confluence of this system (*e.g.* for the sub-system with none of these additional rules).

Then, we present the usual transformations on these derivations from sequent calculus: cut-elimination, rule commutations, and axiom-expansion (Section 1.2).

1.1 Sequent Calculus

1.1.1 Formulas & Orthogonality

Assume given a countable set of **atoms**. Formulas of linear logic are given by the following grammar, where X is an atom:

$$\begin{aligned}
A, B ::= &\mid X^+ \mid X^- && \text{(atom)}\\
&\mid A \parr B \mid A \otimes B \mid \bot \mid 1 && \text{(multiplicative)}\\
&\mid A \,\&\, B \mid A \oplus B \mid \top \mid 0 && \text{(additive)}\\
&\mid ?A \mid !A && \text{(exponential)}\\
&\mid \forall X A \mid \exists X A && \text{(quantifier)}
\end{aligned}$$

We define on formulas the **orthogonality** function $(\cdot)^\perp$ (*a.k.a.* negation or duality) by induction:

$$\begin{aligned}
(X^+)^\perp &= X^- & (X^-)^\perp &= X^+\\
(A \parr B)^\perp &= B^\perp \otimes A^\perp & (A \otimes B)^\perp &= B^\perp \parr A^\perp\\
\bot^\perp &= 1 & 1^\perp &= \bot\\
(A \,\&\, B)^\perp &= B^\perp \oplus A^\perp & (A \oplus B)^\perp &= B^\perp \,\&\, A^\perp\\
\top^\perp &= 0 & 0^\perp &= \top\\
(?A)^\perp &= !A^\perp & (!A)^\perp &= ?A^\perp\\
(\forall X A)^\perp &= \exists X A^\perp & (\exists X A)^\perp &= \forall X A^\perp
\end{aligned}$$

It can be easily checked that orthogonality is an involution: $A^{\perp\perp} = A$.

Nomenclature We call a !-formula one of the shape $!A$, and similarly for a ?-formula, a $\otimes$-formula, etc. The **units** are the formulas 1, $\bot$, 0 and $\top$.

1.1.2 Sequents & Inference rules

Sequents are *sets of (occurrences of) formulas* written in the form $\vdash A_1, \ldots, A_n$. This means we consider only *one-sided* sequents. Everything in this paper should expand trivially to *two-sided* sequents; however, the two-sided framework has twice the number of rules, and thus roughly double the number of cases in our proofs, which already have a considerable number of cases.

Sequent calculus rules of linear logic are given on Figure 3. In these rules, A and B stand for arbitrary formulas, Γ and Δ for contexts (*i.e.* sets of occurrences of formulas). The notation $?\Gamma$ means that each formula of this context is a ?-formula, *i.e.* $?\Gamma = ?A_1, \ldots, ?A_n$. The notation $[B/X]$ is the usual substitution of an atom X by a formula B: each occurrence of X^+ is replaced by B, and each occurrence of X^- by $B^\perp$. The side condition in the application of the $\forall$-rule, X not free (or X fresh) in Γ, is the usual one when handling universal quantifiers: the atom X should not appear in formulas of Γ, whether as X^+ or X^-, except possibly below a quantifier on X, namely $\forall X$ or $\exists X$. This condition can always be satisfied up to α-conversion (renaming of bound variables).

Identity rules:

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

Multiplicative rules:

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

Additive rules:

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

Exponential rules:

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?_d) \qquad \frac{\vdash ?A_1, ?A_2, \Gamma}{\vdash ?A_1, \Gamma}\ (?_c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?_w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

Quantifier rules:

$$X \text{ not free in } \Gamma\ \frac{\vdash A, \Gamma}{\vdash \forall X A, \Gamma}\ (\forall) \qquad B\ \frac{\vdash A[B/X], \Gamma}{\vdash \exists X A, \Gamma}\ (\exists)$$

Optional multiplicative rules:

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ (mix_2) \qquad \frac{}{\vdash}\ (mix_0)$$

Optional additive rules:

$$\frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\ (\cup) \qquad \frac{}{\vdash \Gamma}\ (\emptyset)$$

Figure 3: Rules of the Sequent Calculus of Linear Logic

Conventions

- One may remark the absence of an exchange rule in Figure 3 compared to the usual definitions in the literature (for instance [Gir11; LL22]). In case sequents are lists, the **exchange rule** is the following, where σ is a permutation:

$$\frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ (ex{-}\sigma)$$

  In most papers, immediately after giving this definition, it is written that the *ex*-rule will be used implicitly in the derivations, and this rule is usually not even named afterwards. In the present paper, we look at some precise properties of the sequent calculus, on which the *ex*-rule adds plenty of complexity. Henceforth, we choose to consider as sequents sets of occurrences of formulas and, when we need to consider different occurrences of a same formula, we do so by adding indices (*a.k.a.* names, locations, addresses): *e.g.* $\top_3 \oplus \bot_6$ is the same formula as $\top_8 \oplus \bot_2$, but not the same occurrence. We do not give an explicit definition of occurrence in this paper. We also need to annotate some $\top$- and $?_c$-rules with occurrences, as in the derivation below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \top_1, \top_2, \top_3}\ (\top_1)}{\vdash ?\top_1, \top_2, \top_3}\ (?_d)}{\vdash ?\top_1, ?\top_2, \top_3}\ (?_d)}{\vdash ?\top_1, ?\top_2, ?\top_3}\ (?_d)}{\vdash ?\top_1, ?\top_3}\ (?_c{-}?\top_3)$$

  Another possibility would have been to use *threads* to follow occurrences in a derivation, which is a standard solution when studying non-well-founded derivations—see *e.g.* [Bae+22].

- To not overcomplicate our proofs, *we will always assume the side condition of a $\forall$-rule is satisfied* without stating each time that we may have to rename some bound variables. As such, we will write no condition at the left of a $\forall$-rule when writing a derivation. This amount to working up to α-equivalence, which could be done formally as described in [Lau21]. It should not be a technical limitation, since there is nothing particular here about bound variables. We mainly chose this convention for the sake of simplicity.

- We put the instanciation of an $\exists$-rule at its left, so as to take into account the case where the bound variable does not appear. Nonetheless, we will almost always leave this label implicit.

- Given a rule, its **main formula** (occurrence) is the one it acts on. For instance, with the notations from Figure 3, the main formula of the $\otimes$-rule is (the occurrence) $A \otimes B$, and the main formula of the !-rule is $!A$. Each rule has a main formula, except the identity rules (*ax*- and *cut*-rules) and the optional ones (mix_2- mix_0- $\cup$- and $\emptyset$-rules) which have none.

- As the $\oplus_1$- and $\oplus_2$-rules are very similar, we will sometimes write a $\oplus_i$-rule, meaning a $\oplus_1$-rule or a $\oplus_2$-rule according to the value of $i \in \{1; 2\}$.

1.2 Transformations of derivations

We define here the usual transformations of derivations in linear logic: cut-elimination, rule commutation, and axiom-expansion. These transformations are *contextual*—they can be applied in a sub-derivation, *i.e.* they are closed by context—and most of them are *local*—they modify only part of a derivation, rules that are above or below stay the same. The tables giving these transformations are quite big, and can be found at the very end of this paper in Appendix E (from Page 99 to Page 119).

1.2.1 Cut-elimination

In most systems with a *cut*-rule, one shows this rule is *admissible*, *i.e.* that the same sequents can be proved with the *cut*-rule than without it. The procedure turning a derivation into a *cut*-free one is called *cut-elimination*, and introduces a notion of computation in the logic.

Definition 1. Cut-elimination is the rewriting system denoted $\xrightarrow{\beta}$ whose rules are described on Tables 2 to 4, up to commuting the two branches of any *cut*-rule. (This means one should take each rule in these tables and also consider a version of this rule with the left and right premises of any *cut*-rule swapped. In particular, for the $cut - cut$ commutative case, there are in fact 4 rewriting rules.)

Remark 2.

- In the $?_c - !$ and $?_w - !$ exponential key cases, the notation of a doubled inference rule means we have to apply the corresponding rule a certain number of times, once on each formula of $?\Delta$. Remark there are several possible orders in which these rules can be applied: two such $?_c - !$ (or two $?_w - !$) steps can be applied on the same rules and result in different derivations.

- The $\forall - \exists$ key case substitutes X by B in the derivation π_1, yielding a well-defined derivation since $A^\perp[B/X] = (A[B/X])^\perp$ is easily provable by induction.

- As stated beforehand, in the $\forall - cut$ commutative case one may need to α-rename the atom X into a new atom Y so as to respect Y not free in Γ, Δ.

When studying properties of cut-elimination, we will need to consider a bipartition of cut-elimination into $\xrightarrow{\beta} = \xrightarrow{\overline{\beta}} \cup \overset{c}{\longmapsto\!\dashv}$.

Definition 3. We denote by $\xrightarrow{\overline{\beta}}$ a $\xrightarrow{\beta}$ step other than a $cut - cut$ commutation. We call $\overset{c}{\longmapsto\!\dashv}$ a $cut - cut$ commutation.

Please notice that $\overset{c}{\longmapsto\!\dashv}$ is a symmetric relation.

1.2.2 Rule commutations

A rule commutation associates two derivations which differ only by the order in which some of their rules are applied. In other words, this relation appears when one can apply two successive rules, say r and s, and both applying r then s or s then r would do the job. That is why rule commutations can be viewed as "bureaucracy" [Gir01], as both choices are adequate, and in fact are not really a choice as we would like to consider both derivations to be the same. But due to the definition of a derivation, we must choose an arbitrary order and apply one of the rules first, for it is impossible to apply two rules at once in sequent calculus.

Definition 4. Rule commutation is the symmetric relation denoted $\overset{r}{\longmapsto\!\dashv}$ whose rules are described on Tables 5 to 20; all sub-derivations named on these tables must be cut-free.

These rule commutations are exactly those given by the following algorithm. Consider a *cut*-rule c and all couples of non *cut*-rules that can be the premises of c, such that two commutative cut-elimination cases can be applied on c. Comparing the results of applying the left commutative step then the right, against applying the right then the left, yields the rule commutations. This is why there is no commutation with an *ax*-, 1-, 0- nor mix_0-rule for there is no associated commutative cut-elimination case for these rules. Also, there is no $\emptyset - \emptyset$ rule commutation as it would be the identity: $\dfrac{}{\vdash \Gamma}\,(\emptyset) \quad \xrightarrow{C^{\emptyset}_{\emptyset}} \quad \dfrac{}{\vdash \Gamma}\,(\emptyset)$.

Please notice the $\exists - \forall$ commutation: it is the sole place in this paper where the side condition of the $\forall$-rule really matters (*i.e.* cannot always be satisfied by α-renaming) since pushing an $\exists$-rule below a $\forall$-rule in a $C^{\exists}_{\forall}$ commutation can only be done when the witness of the $\exists$-rule does not depend on the variable introduced by the $\forall$-rule.

Remark 5. Definition 4 corresponds to rule commutations in cut-free linear logic, *i.e.* with no *cut*-rule above the commutation, whereas there may be *cut*-rules below the commutation, in the context. In particular, in a $\top - \otimes$ (resp. $\& - \otimes$) commutation we assume the created or erased sub-derivation (resp. duplicated or superimposed sub-derivation) to be cut-free. This choice is more appropriate for our setting because it preserves cut-freeness. A more general theory of rule commutation exists, including the *cut*-rule: see [HG16] in the case of multiplicative-additive linear logic without units and with the mix_2-rule.

As for cut-elimination, we will need to spearate in two categories these rule commutations.

Definition 6. Set $\xrightarrow{\top}$ the rule commutations involving $\top$- or $\emptyset$-rules oriented in $C^{\top}_{-}$ and $C^{\emptyset}_{-}$ in Tables 5 to 20, and $\overset{r\setminus\top}{\longmapsto\!\!\dashv}$ the rule commutations not involving a $\top$- nor $\emptyset$-rule.

1.2.3 Additional transformations

There exist other transformations on derivations in linear logic, that one may or may not want to include in equality of derivations. Those concern the exponential rules and the comonoidal character of ?, as well as the two pairs of optional rules. One may want (or not) that:

- $?_c$- and $?_w$-rules commute with the !-rule;
- $?_c$ is coassociative and cocommutative, and $?_w$ is coneutral for $?_c$;
- mix_2 is cocommutative, and mix_0 is coneutral for mix_2;
- $\cup$ is cocommutative, and $\emptyset$ is coneutral for $\cup$.

(Observe that coassociativity of mix_2 and of $\cup$ are already in rule commutation.)

Indeed, to obtain the famous Seely isomorphism $!(A \,\&\, B) \simeq\ !A \otimes !B$ one uses the first two items. The third item is a quotient made by proof-nets (see *e.g.* [Di+25]), and the fourth item is its additive counterpart.

Definition 7. The **Rétoré reduction** is the relation denoted $\xrightarrow{R}$ and defined as the union of $\xrightarrow{Re}$, $\xrightarrow{Rm}$ and $\xrightarrow{Ra}$ whose rules are described on Table 21.

When considering $\xrightarrow{R}$, one can choose to take (or not) any of the $\xrightarrow{Re}$, $\xrightarrow{Rm}$ and $\xrightarrow{Ra}$ rules, leading to $2^3 = 8$ different systems.

1.2.4 Axiom-expansion

As in many logics with an ax-rule, one may consider derivations with the ax-rule reduced to the case where it is applied on an atom, *i.e.* with $A = X^+$:

$$\dfrac{}{\vdash X^-, X^+}\ (ax)$$

Such derivations can be obtained through a rewriting procedure named axiom-expansion.

Definition 8. **Axiom-expansion** is the rewriting system denoted $\xrightarrow{\eta}$ whose rules are described on Table 22.

2 Rewriting theory modulo an equivalence relation

We define here concepts from the theory of abstract rewriting systems, and state the result we will use to prove that cut-elimination is Church-Rosser modulo rule commutation. Most of the definitions and notations from this section are taken from [Ter03].

We use standard notations from relation algebra and rewriting theory. The composition of relations is denoted by $\cdot$, their union by $\cup$ and their inclusion by $\subseteq$. Given a relation $\xrightarrow{\alpha}$ on some set, we write $\xrightarrow{\alpha^*}$ (resp. $\xrightarrow{\alpha^+}$, resp. $\xrightarrow{\alpha^=}$) the transitive reflexive (resp. transitive, resp. reflexive) closure of $\xrightarrow{\alpha}$, and $\xleftarrow{\alpha}$ the converse relation—symmetric relations will correspond to symmetric symbols—; the equivalence closure of $\xrightarrow{\alpha}$, *i.e.* $(\xrightarrow{\alpha} \cup \xleftarrow{\alpha})^*$, is denoted by $=_\alpha$. Below are recalled the definitions of normalization and confluence.

Definition 9 (Normalization, Confluence)**.** Let $\to$ be a binary relation on a set A.

- Given $a, b \in A$ such that $a \to b$ we say a **reduces** to b.
- An element $a \in A$ is **a $\to$-normal form** if there exists no $b \in A$ such that $a \to b$. If $a \to^* b$ with b a $\to$-normal form, we say b is **a $\to$-normal form of** a.
- The relation $\to$ is **weakly normalizing** if for all $a \in A$, there exists a $\to$-normal form $b \in A$ such that $a \to^* b$—*i.e.* if there exists a $\to$-normal form of any element.
- The relation $\to$ is **strongly normalizing** if for all $a \in A$, any rewriting sequence starting from a is finite, *i.e.* of the shape $a \to^* b$ for some $b \in A$.
- The relation $\to$ is **confluent**, or **Church-Rosser**, if ${}^*\!\leftarrow \cdot \to^* \subseteq \to^* \cdot {}^*\!\leftarrow$.

Remark that strong normalization implies weak normalization. As indicated in the introduction, cut-elimination in linear logic is *not* confluent. Nonetheless, we will prove it is Church-Rosser modulo rule commutation, from which we will deduce our main result, Theorem 22.

Definition 10 (Church-Rosser modulo)**.** Let $\sim$ and $\to$ be relations on a set A such that $\sim$ is an equivalence relation. The relation $\to$ is *Church-Rosser modulo* $\sim$ if $(\to \cup \leftarrow \cup \sim)^* \subseteq \to^* \cdot \sim \cdot {}^*\!\leftarrow$ (see Figure 4).

There is in the literature a variety of results proving a Church-Rosser modulo property, *e.g.* [Hue80; JK84; Oos94; Ohl98; AT12; Fel24]. Some can be seen as a generalization of the well-known Newman's Lemma [Ter03, Theorem 1.2.1] to prove Church-Rosser modulo instead of confluence. Unfortunately, most of those results cannot be applied to cut-elimination of linear logic. In particular, none of the three theorems from [Ter03, Section 14.3]—two due to Huet [Hue80, Lemmas 2.7, 2.8] and one to van Oostrom [Oos94, Proposition 2.5.3]—can be

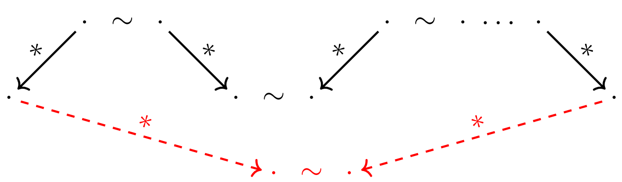


Figure 4: Diagram of Church-Rosser modulo (Definition 10), with hypotheses in solid black and conclusions in dashed red

applied in our case. First, results that consider closing diagrams with as hypotheses $a \overset{*}{\vdash\!\dashv} b$ (*e.g.* coherence with an equivalence relation) can hardly be used in our case. This is because being equal up to an arbitrary number of rule commutations is too complex and cannot really be exploited, except by looking at the rule commutations one by one—remember we will prove that equality up to rule commutation in undecidable in Section 8. This is why we cannot apply *e.g.* [Hue80, Lemma 2.7; Oos94, Proposition 2.5.3; Ohl98, Lemma 7]. While doing cases on a step of $\overset{r}{\vdash\!\dashv}$ is as tedious as it is boring, it still has the advantage of being a simple case study. Second, many results that consider only $\vdash\!\dashv$ and not $\overset{*}{\vdash\!\dashv}$ (*e.g.* using local coherence and not coherence) require the strong normalization of $\rightarrow \cdot \overset{*}{\vdash\!\dashv}$, which is false for cut-elimination and rule commutations—see Section 3.1 for a counter-example. This prevents us from applying *e.g.* [Hue80, Lemma 2.8; JK84, Theorem 1]. What we will use is the following.

Proposition 11. *Let $\rightarrow$, $\vdash\!\dashv$ and $\rightsquigarrow$ be relations on a set A such that $\vdash\!\dashv$ is symmetric and $\rightsquigarrow \subseteq \vdash\!\dashv$. Set $\Rightarrow := \rightarrow \cup \rightsquigarrow$. Suppose:*

(i) *$\rightarrow \cdot \rightsquigarrow^*$ is strongly normalizing;*

(ii) *$\leftarrow \cdot \rightarrow \subseteq \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$;*

(iii) *given $a, b, c \in A$, if $a \vdash\!\dashv b \rightarrow c$ then $a \rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow \cdot \leftarrow b$.*

Then $\rightarrow$ is Church-Rosser modulo $\overset{*}{\vdash\!\dashv}$.

Proof. The relation $< := (\rightsquigarrow^* \cdot \rightarrow \cdot \rightsquigarrow^*)^*$ is a well-founded strict partial order on the set A: transitivity is immediate, and irreflexivity and well-foundedness come from the strong normalization of $\rightarrow \cdot \rightsquigarrow^*$. Take a sequence ρ of elements of A related by either $\vdash\!\dashv$, $\rightarrow$ or $\leftarrow$. Pose the multiset $\mathcal{M}(\rho) := \biguplus_{a \vdash\!\dashv b \,\in\, \rho}\{a, a, b, b\} \uplus \biguplus_{a \rightarrow b \,\in\, \rho}\{a\} \uplus \biguplus_{a \leftarrow b \,\in\, \rho}\{b\}$, equipped with a well-founded strict partial order $<$ by the multiset extension of the one on A. We proceed by induction on $\mathcal{M}(\rho)$.

If ρ is of the shape $\rightarrow^* \cdot \overset{*}{\vdash\!\dashv} \cdot {}^*\!\leftarrow$ then we are done. Otherwise, ρ contains a sub-sequence $a \leftarrow b \rightarrow c$, or $a \vdash\!\dashv b \rightarrow c$, or $c \leftarrow b \vdash\!\dashv a$, leading to a case study.

Assume ρ contains a sub-sequence $a \leftarrow b \rightarrow c$. We replace it with a sub-sequence $a \Rightarrow \alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_n \overset{=}{\vdash\!\dashv} \alpha_{n+1} \Leftarrow \cdots \Leftarrow c$. The resulting sequence ρ' is such that $\mathcal{M}(\rho') < \mathcal{M}(\rho)$, allowing to conclude by induction hypothesis. Indeed, our operation replaces in the measure the multiset $\{b, b\}$ by a multiset upper-bounded by $\{a, a, c, c\} \uplus \biguplus_i \{\alpha_i, \alpha_i, \alpha_i, \alpha_i\}$ with, for $i \leq n$, $\alpha_i \leq a < b$ and, for $i > n$, $\alpha_i \leq c < b$.

Suppose now ρ contains a sub-sequence $a \vdash\!\dashv b \rightarrow c$ or $c \leftarrow b \vdash\!\dashv a$. We replace the sub-sequence $a \vdash\!\dashv b$ by $a \rightarrow \alpha_1 \Rightarrow \cdots \Rightarrow \alpha_n \overset{=}{\vdash\!\dashv} \alpha_{n+1} \Leftarrow \cdots \Leftarrow \alpha_m \leftarrow b$, yielding a new sequence ρ'. This removes from $\mathcal{M}(\rho)$ the multiset $\{a, a, b, b\}$, and adds at most $\{a, b\} \uplus \biguplus_i \{\alpha_i, \alpha_i, \alpha_i, \alpha_i\}$. We conclude by induction again since $\mathcal{M}(\rho') < \mathcal{M}(\rho)$ as, for $i \leq n$, $\alpha_i < a$ and, for $i > n$, $\alpha_i < b$. $\square$

$$\pi_1 := \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash ?\bot_1, \top}\,(\top)}{\vdash ?\bot_1, ?\bot_2, \top}\,(?_w)}{\vdash ?\bot_1, \top}\,(?_c)}{\vdash \top}\,(\mathit{cut}) \quad \overset{r}{\vdash\!\dashv} \quad \pi_4 := \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{}{\vdash ?\bot_1, ?\bot_2, \top}\,(\top)}{\vdash ?\bot_1, \top}\,(?_c)}{\vdash \top}\,(\mathit{cut}) \quad \overset{r}{\vdash\!\dashv} \quad \pi_3 := \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{}{\vdash ?\bot_1, \top}\,(\top)}{\vdash \top}\,(\mathit{cut})$$

$$\pi_1 \xrightarrow{\overline{\beta}} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$$

$$\pi_2 := \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{}{\vdash ?\bot_1, \top}\,(\top)}{\vdash ?\bot_1, ?\bot_2, \top}\,(?_w)}{\vdash ?\bot_1, \top}\,(\mathit{cut})}{\vdash \top}\,(\mathit{cut})$$

Figure 5: Counter-example to the strong normalization of $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$

This result is heavily inspired by a theorem from Aoto and Toyama [AT12, Theorem 2.2]. In particular, our proof of Proposition 11 is almost identical to the one of [AT12, Lemma 2.1], which implies [AT12, Theorem 2.2]. Please notice that in (Proposition 11).(iii), the only use of the variable c is to ensure that b is not a $\to$-normal form. This means that the found sequence $a \to \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow \cdot \leftarrow b$ may end with $c \leftarrow b$, or with $d \leftarrow b$ for some $d \neq c$. Intuitively, (Proposition 11).(ii) can be seen as a generalization of local confluence (*a.k.a.* property α) [Ter03, Definition 14.3.3] and (Proposition 11).(iii) as a generalization of local coherence (*a.k.a.* property γ) [Ter03, Definition 14.3.3].

We will instantiate Proposition 11 with $\to := \xrightarrow{\overline{\beta}}$, $\vdash\!\dashv := \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}$, and $\rightsquigarrow := \overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top}$. We will first prove the needed strong normalization result in Section 3, and then show the two other hypotheses of Proposition 11 in Section 4. This yields that $\xrightarrow{\overline{\beta}}$ is Church-Rosser modulo $(\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv})^*$, which easily implies that $\xrightarrow{\beta}$ is Church-Rosser modulo $\overset{r^*}{\vdash\!\dashv}$.

3 Strong normalization up to rule commutation

In linear logic, $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$ is *not* strongly normalizing which, as stated in Section 2, prevents many theorems about Church-Rosser modulo from being applied to our case. Nonetheless, we prove the strong normalization of $\xrightarrow{\overline{\beta}} \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top})^*$, *i.e.* with some rule commutations *oriented*, which is enough to apply Proposition 11. We begin by providing a simple counter-example for the strong normalization of $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$ (Section 3.1), and then state and demonstrate our main result on strong normalization thanks to [PT10] (Section 3.2).

3.1 No strong normalization up to rule commutation

Figure 5 presents a simple counter-example to the strong normalization of $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$. Using successively a $?_c - !$ and a $?_w - !$ key cut-elimination cases, one gets $\pi_1 \xrightarrow{\overline{\beta}} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$; then, through a $\top - ?_c$ commutation followed by a $\top - ?_w$ commutation, $\pi_3 \overset{r}{\vdash\!\dashv} \pi_4 \overset{r}{\vdash\!\dashv} \pi_1$. Thus, $\pi_1 \xrightarrow{\overline{\beta}^+} \cdot \overset{r^+}{\vdash\!\dashv} \pi_1$, where those cut-elimination steps are key cases!

There are other counter-examples to the strong normalization of $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$, involving $\top - \otimes$ or $\top - \exists$ or $\top - \mathit{mix}_2$ or $\top - \cup$ commutations—see Appendix A for those.

3.2 Strong normalization up to oriented rule commutation

We give here our main result about normalization, Theorem 12: strong normalization holds *up to orienting* some rule commutations involving $\top$- and $\emptyset$-rules; namely $\xrightarrow{\overline{\beta}} \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top} \cup \xrightarrow{R})^*$ is strongly normalizing. This almost follows from a journal paper by Michele Pagani and Lorenzo Tortora de Falco [PT10], whose main result is strong normalization for cut-elimination in proof-nets of linear logic with second order quantifiers. The aforementioned paper is a technical one of 35 dense pages, involving non-trivial reasonings about rewriting theory and non-standard proof-nets, called *sliced pure structures* [PT10, Definition 2.2]. These are graphs equipped with a reduction procedure $\longrightarrow$ that is strongly normalizing (provided some typing conditions) [PT10, Theorem 4.2]. The idea underlying our proof is a simple adaptation of the one of [PT10]: there is a translation $\mathcal{T}$ from derivations of sequent calculus to sliced pure structures, with the following properties.

- The images of $\mathcal{T}$ are easily shown to be strongly normalizing.
- If $\pi \xrightarrow{\overline{\beta}} \phi$ then $\mathcal{T}(\pi) \xrightarrow{*} \mathcal{T}(\phi)$.
- If $\pi \overset{c}{\vdash\!\dashv} \phi$ or $\pi \overset{r\setminus\top}{\vdash\!\dashv} \phi$ then $\mathcal{T}(\pi) = \mathcal{T}(\phi)$.
- If $\pi \xrightarrow{\top} \phi$ or $\pi \xrightarrow{R} \phi$ then $\mathcal{T}(\pi) \xrightarrow{*} \mathcal{T}(\phi)$ using some *new* reduction rules.

The strong normalization of $\xrightarrow{\overline{\beta}} \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top} \cup \xrightarrow{R})^*$ in derivations then easily follows.

Nonetheless, our result is not formaly a corollary of [PT10, Theorem 4.2], but of a generalization of it with a few more reduction rules (corresponding to the $\xrightarrow{\top}$ rule commutations) and a slightly modified definition of sliced pure structures (to have invariance by a $\top - \top$ commutation). As this definition is at the core of [PT10], and occurs at the very beginning of this paper, one needs to check that every following definition and statement in [PT10] still holds with almost the same proof. Remarkably, most of the proofs in [PT10] can be almost directly transposed as proofs for our slightly different definitions, so that we mainly introduce new easy cases. As we do not wish to rewrite almost verbatim this paper in full, nor do we think most readers would be interested in the resulting text, we simply explain here what are the main modifications to apply to [PT10] in order to obtain the wanted result—precise statements for intermediate claims are given in Appendix B, whose proof a motivated reader could easily find starting from the ones in [PT10].

Theorem 12. *The relation* $(\xrightarrow{\overline{\beta}} \cup \xrightarrow{R}) \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top})^*$ *is strongly normalizing, which in particular implies that* $\xrightarrow{\overline{\beta}} \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top} \cup \xrightarrow{R})^*$ *is strongly normalizing.*

Sketch of a proof. We assume here that the reader has a copy of [PT10] at their disposition, and that they are vaguely familiar with some concepts of linear logic such as proof-nets [Gir96] and slices [Gir87]. If it is not the case, this sketch of proof will surely be unintelligible since we cannot afford to reproduce here all intuitons and explanations from [PT10]. The precise intermediate statements for this proof are given in Appendix B, without proofs since they are the same as those of the corresponding results in [PT10] up to some easy new cases.

The idea is to adapt [PT10, Theorem 5.12], which is about strong normalization for (a kind of) proof-nets for linear logic with second order quantifiers, to derivations of the sequent calculus defined in Section 1. The main technical considerations are in the proof of [PT10, Theorem 4.2], which is the most technical result of that paper. As the whole point of that long paper is to prove this theorem, checking our modifications necessitates many steps. This extended sketch presents what the modifications are, and which proofs are to be adapted—very often in a trivial way. As indicated at the end of the introduction of [PT10] (in the "Added in print" paragraph), we do not have to consider Section 3 of that paper, as their main result [PT10, Theorem 4.2] can be proved through weaker hypotheses than what is written.

The authors of [PT10] consider *sliced pure structures* [PT10, Definition 2.2], which are basically proof-nets that are only correct slice by slice, called *AC correctness*. A small modification in their definition is needed in our case: a $\top$-link should not have any distinguished (or main) conclusion but only auxiliary conclusions—this requires adapting the definition of a flat [PT10, Definition 2.1, Figure 2]. Remark that, with this new definition, a $\top$-link may have zero conclusion. This modification makes sliced pure structures invariant by $\top - \top$ commutations, and also allows to interpret a $\emptyset$-rule by a $\top$-link and having a definition invariant by $\top - \emptyset$ commutations.

Considering reduction rules, this slight modification implies some changes for the definition of cut-elimination in sliced pure structures [PT10, Definition 2.12]. Morevoer, a key difference is that we also add to this definition of reduction rules new (or more general) cases, corresponding to $\xrightarrow{\top}$, $\xrightarrow{Ra}$ and $\xrightarrow{Re}$ on sliced pure structures. There is no need to add a rule corresponding to $\xrightarrow{Rm}$ because sliced proof structures already quotient by this relation.

- First, consider the (ax) rewriting rule from [PT10, Definition 2.12]. When eliminating a *cut* below an *ax*, we do not need to assume the other premise of the *cut* is not the auxiliary conclusion of a $\top$. Now, there might be two possible cut-elimination cases applicable on a given *cut*, if it has above its premises an *ax*-link and a $\top$-link, but this is quite harmless—there is still some basic verification to do, see [PT10, Remark 2.13].

- We generalize the $(\top/\mathrm{cc})$ rewriting rule from [PT10, Definition 2.12] as follows: given a slice containing a $\top$-link l and a module γ not containing l and whose hypotheses are all conclusions of l, we replace l and γ by a new $\top$-link with as conclusions the conclusions of γ and the conclusions of l that are not hypotheses of γ. Intuitively, this step not only generalizes both $\top - cut$ and $\emptyset - cut$ reductions, but also contains $\xrightarrow{\top}$. Such a reduction may not be the image of one from the sequent calculus, but it is already the case for other reductions in [PT10]—including the original $(\top/\mathrm{cc})$ rewriting rule—and it does not matter here as strong normalization for more steps implies strong normalization for the wished steps.

- We generalize the $(\oplus_i/\&_j)$ rewriting rule from [PT10, Definition 2.12], with $i \neq j$, as follows: given a sliced pure structure α and $s \in \alpha$ one of its slices, we reduce it to $\alpha\backslash s$, *i.e.* the multiset α where s is removed. This generalizes both the two $\oplus_i - \&_j$ reduction rules and the $\xrightarrow{Ra}$ reduction.

- We add a new rewriting rule, corresponding to $\xrightarrow{Re}$. This is the usual transformation $\xrightarrow{Re}$ on MELL proof-nets [DR95; Reg92]: in a slice, a $?_w$-link above a $?_c$-link can be simplified by removal of these two vertices.

One can then follow all results of the paper, and see its proofs still hold after our modifications. First, that [PT10, Proposition 2.16] still holds is trivial. We classify the new $\xrightarrow{Re}$ case as an erasing step [PT10, Definitions 4.1], which is also the class of the modified $(\top/\mathrm{cc})$ and $(\oplus_i/\&_j)$ cases. Erasing steps are considered only in the beginning of the proof of [PT10, Theorem 4.2] (in Section 4 of that paper, remember we do not need to consider its Section 3), hence only a few results are to be checked: these are [PT10, Proposition 4.5] and two intermediate results to prove it, namely [PT10, Lemmas 4.3 and 4.4] (the first lemma being only used to prove the second). Here, we need a more general notion of *postponement* than in that paper [PT10, Figure 4(e)]: *a relation* $\xrightarrow{x}$ *can be postponed with respect to a relation* $\xrightarrow{y}$ *when for every* π, π_1, π_2 *such that* $\pi \xrightarrow{x} \pi_1 \xrightarrow{y} \pi_2$, *there is* π_3 *such that* $\pi \xrightarrow{y} \pi_3 \xrightarrow{(x \cup y)^*} \pi_2$. One can check that [PT10, Lemma 4.4] still holds after our modifications with this new definition of postponement, and that [PT10, Proposition 4.5] follows with the very same proof as written in that paper with this more general postponement. The main difference in the proof of Lemma 4.4 is for a $\xrightarrow{Re}$ step: to postpone such a step with respect to a non-erasing step, in the case of a cut-elimination step between a *cut* below the erased $?_c$-link and the link above the latter, one has to apply a non-erasing $?_c - !$ key case, then the considered non-erasing cut-elimination case, then an erasing $(!/?_w)$ step—hence our need for a more general postponement. Therefore, the adaptation of [PT10, Theorem 4.2] is true for our slightly different sliced pure structures: a sliced pure structure satisfying AC (acyclicity by slice) which is weakly normalizing for non-erasing steps is strongly normalizing for all steps, including the new erasing ones we just added.

We now adapt [PT10, Section 5] by considering derivations of sequent calculus instead of proof-nets. This is quite easy, as the authors of [PT10] consider proof-nets as the images of derivations: given a proof-net β, the associated sliced pure structure $sl(\beta)$ is defined from a derivation π desequentializing to β. We thus simply define the sliced pure structure $sl(\pi)$ associated to a derivation π as done in [PT10, Section 5.3], with the obvious extension for the optional rules:

- a mix_2-rule is translated as the union of all couples of slices obtained from its two premises (similarly to a $\otimes$-rule);
- a mix_0-rule is translated as the empty slice;
- a $\cup$-rule is translated as taking the multiset union of the slices (similarly to a &-rule);
- a $\emptyset$-rule is translated by a single slice consisting of a $\top$-link (similarly to a $\top$-rule).

It is easy and standard to prove by induction that the image of a derivation satisfies AC correctness, yielding a parallel of [PT10, Proposition 5.1] for derivations instead of proof-nets. Also, the results of [PT10, Section 5.4], namely that typed AC sliced pure structures are weakly normalizing for non-erasing steps, is still true in this setting: an adaptation of its proof by replacing proof-nets with derivations is enough to get that if π is a derivation then $sl(\pi)$ is weakly normalizing for non-erasing steps, which is [PT10, Theorem 5.11]. In particular, the authors of [PT10] proceed here by means of reducibility candidates for proof-nets, that we can replace by the obvious notion of reducibility candidates for derivations of sequent calculus, and adapting the proof of [PT10, Theorem 5.11] presents no challenge since the authors proceed by an induction on a sequentialization of a proof-net, *i.e.* directly at the level of sequent calculus.

Afterwards starts the core of this proof, where considering derivations instead of proof-nets makes a real difference. One has to prove equivalents of [PT10, Lemmas 5.3, 5.4, 5.5, Proposition 5.6] in the framework of sequent calculus, namely:

- If $\pi \xrightarrow{\overline{\beta}} \pi'$ or $\pi \xrightarrow{\top} \pi'$ or $\pi \xrightarrow{R} \pi'$, then $sl(\pi) \xrightarrow{*} sl(\pi')$ (remember we put the steps corresponding to $\xrightarrow{\top}$ and $\xrightarrow{R}$ in sliced pure structures), with $sl(\pi) = sl(\pi')$ if and only if this cut-elimination step is a $\forall - \exists$ key step, or is a commutative step other than $\top - cut$, $! - cut$ and $\emptyset - cut$, or if the step is a $\xrightarrow{Rm}$ step, or if the step is a $\top - \top$ or $\top - \emptyset$ commutation.
- If $\pi \overset{r\setminus\top}{\longmapsto} \pi'$ then $sl(\pi) = sl(\pi')$.
- If $\pi \overset{c}{\longmapsto} \pi'$ then $sl(\pi) = sl(\pi')$.

We then conclude by contradiction. Taking an infinite sequence of $(\xrightarrow{\overline{\beta}} \cup \xrightarrow{R}) \cdot (\overset{c}{\longmapsto} \cup \overset{r\setminus\top}{\longmapsto} \cup \xrightarrow{\top})^*$ reduction steps in sequent calculus, its image in sliced pure structures cannot be an infinite sequence of reduction steps since we proved strong normalization. Therefore, there exists an infinite suffix of this sequence where all $\xrightarrow{\overline{\beta}}$, $\xrightarrow{\top}$ and $\xrightarrow{R}$ steps are those preserving images by $sl(\cdot)$: those are $\forall - \exists$ key steps or commutative steps other than $\top - cut$, $! - cut$ and $\emptyset - cut$, $\top - \top$ or $\top - \emptyset$ commutations, or $\xrightarrow{Rm}$ steps. We claim only a finite number of such steps can be applied, including when interleaving them with $(\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv})^*$ steps. Call a *block* of *cut*-rules of root c in a given derivation π of sequent calculus a maximal set of *cut*-rules in π which are all premises of the *cut*-rule c, or premises of another *cut*-rule in this set. We show each step in the considered suffix decreases the following measure on derivations: $\sum_r \sum_s (n(r,s)+1)$ with this sum being over all rules r in the derivation and all slices s containing r, and $n(r,s)$ being the number of *cut*-rules in blocks situated below r in the slice s (*i.e.* blocks of root c with c closer to the root of the slice than r). Observe that a $\forall - \exists$ key step decreases this quantity because it removes rules, as does a $\xrightarrow{Rm}$ step, and as does a commutative step which is not a $\top - cut$, $! - cut$, $\emptyset - cut$ nor a $cut - cut$ step—since it pushes some rule(s) below a *cut*-rule. Moreover, a $\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv}$ step preserves this quantity since it preserves the number of slices, the number and kind of rules by slices and the blocks of *cut*-rules per slice. It is also preserved by $\top - \top$ and $\top - \emptyset$ commutations. We conclude that such an infinite suffix cannot exist: $(\xrightarrow{\overline{\beta}} \cup \xrightarrow{R}) \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top})^*$ is strongly normalizing. ☐

An immediate corollary of Theorem 12, that will be of use afterwards, is the well-known weak normalization of cut-elimination.

Corollary 13. *Cut-elimination with and without the $cut - cut$ commutation, $\xrightarrow{\beta}$ and $\xrightarrow{\overline{\beta}}$, are both weakly normalizing. Moreover, cut-free derivations are exactly $\xrightarrow{\beta}$-normal forms and exactly $\xrightarrow{\overline{\beta}}$-normal forms.*

Proof. Using Theorem 12, any derivation π has a $\xrightarrow{\overline{\beta}}$-normal form ϕ. But derivations in $\xrightarrow{\overline{\beta}}$-normal form correspond to cut-free derivations: as long as there is a *cut*-rule, a $\xrightarrow{\overline{\beta}}$ step can be applied. Thus, no $\overset{c}{\vdash\!\dashv}$ step can be applied on ϕ: it is also a $\xrightarrow{\beta}$-normal form of π. ☐

4 Church-Rosser modulo for cut-elimination

We prove here our key intermediate result: cut-elimination $\xrightarrow{\beta}$ is Church-Rosser modulo rule commutation $\overset{r^*}{\vdash\!\dashv}$. We obtain it by showing $\xrightarrow{\overline{\beta}}$ is Church-Rosser modulo $(\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv})^*$, thanks to an application of Proposition 11 instanciated with $\rightarrow := \xrightarrow{\overline{\beta}}$, $\vdash\!\dashv := \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}$, and $\rightsquigarrow := \overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top}$. As the hypothesis about strong normalization, (Proposition 11).(i), has already been proved in the previous section, we have to show (Proposition 11).(ii) and (Proposition 11).(iii). This means we have to study $\xleftarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}}$ (Section 4.1), $\overset{c}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}}$ (Section 4.2) and $\overset{r}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}}$ (Section 4.3), so as to conclude (Section 4.4).

In this section we denote graphically some derivations with the following convention. When writing derivations as in

$$\dfrac{\dfrac{\rho}{\quad}\,(r_2)}{\quad}\,(r_1) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\rho}{\quad}\,(r_1)}{\quad}\,(r_2)$$

we abuse notations in the cases where r_1 or r_2 is not a unary rule. The meaning is that, if say r_1 is a &-rule, then r_2 is duplicated, and even possibly a whole sub-derivation if r_2 is a $\otimes$-rule for instance. Similarly, if r_1 is a $\top$-rule, then this schema means that on the left hand-side r_2 and ρ are not here, and are created by the $\top$-commutation.

4.1 Generalized local confluence

We prove here our instanciation of (Proposition 11).(ii).

Lemma 14. *Let π, π_1 and π_2 be derivations such that $\pi_1 \xleftarrow{\overline{\beta}} \pi \xrightarrow{\overline{\beta}} \pi_2$. Then $\pi_1 \overset{c=}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}^*} \cdot (\overset{r\setminus\top^*}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}) \cdot \xleftarrow{\overline{\beta}^*} \pi_2$. Diagrammatically:*

$$\begin{tikzcd}
 & \pi \arrow[dl, "\overline{\beta}"'] \arrow[dr, "\overline{\beta}"] & \\
\cdot \overset{c=}{\vdash\!\dashv} \pi_1 \arrow[dr, dashed, "\overline{\beta}^*"'] & & \pi_2 \arrow[dl, dashed, "\overline{\beta}^*"] \\
 & \cdot \overset{r\setminus\top^*}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv} \cdot &
\end{tikzcd}$$

Proof. A first easy case is when the $\pi \xrightarrow{\overline{\beta}} \pi_1$ and $\pi \xrightarrow{\overline{\beta}} \pi_2$ steps involve only distinct rules. Then, taking into account that rules of one step may be duplicated or erased by the other step, they commute and we have a derivation π' such that $\pi_1 \xrightarrow{\overline{\beta}^*} \pi' \xleftarrow{\overline{\beta}^*} \pi_2$, by applying one reduction after the other. With more details:

- $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xleftarrow{\overline{\beta}} \pi_2$ if neither step duplicates nor erases the other;
- $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \cdot \xleftarrow{\overline{\beta}} \pi_2$ if $\pi \xrightarrow{\overline{\beta}} \pi_1$ duplicates a sub-derivation containing the rules involved in $\pi \xrightarrow{\overline{\beta}} \pi_2$, which may happen if $\pi \xrightarrow{\overline{\beta}} \pi_1$ is a $?_c - !$ key case or a $\& - cut$ or $\cup - cut$ commutative case (and symmetrically if we swap indices 1 and 2);
- $\pi_1 \xleftarrow{\overline{\beta}} \pi_2$ if $\pi \xrightarrow{\overline{\beta}} \pi_1$ erases a sub-derivation containing the rules involved in $\pi \xrightarrow{\overline{\beta}} \pi_2$, which may happen if $\pi \xrightarrow{\overline{\beta}} \pi_1$ is a $\& - \oplus_i$ or $?_w - !$ key case or a $\top - cut$ or $\emptyset - cut$ commutative case (and symmetrically if we swap indices 1 and 2).

There is no other case, as to duplicate or erase rules a $\xrightarrow{\overline{\beta}}$ step must be applied on rules below those. In particular, both steps cannot duplicate or erase the rules of the other step, because the rules involved are distinct.

Consider now the other easy case where both steps have all of their rules in common. If the two reductions are not both $⅋ - \otimes$ nor both $?_c - !$ nor both $?_w - !$ key cases, then the two reductions are the same: $\pi_1 = \pi_2$ and we are done. If both reductions are $⅋ - \otimes$ key cases on the same rules, then the two resulting derivations are either equal or related by $\overset{c}{\vdash\!\dashv}$: $\pi_1 \overset{c=}{\vdash\!\dashv} \pi_2$. If both reductions are $?_c - !$ (resp. $?_w - !$) key cases on the same rules, then the two resulting derivations are equal up to $?_c - ?_c$ (resp. $?_w - ?_w$) rule commutations on the produced rules, and maybe also up to $\overset{c}{\vdash\!\dashv}$ to commute the two produced *cut*-rules. As we only apply rule commutations on cut-free sub-derivations, we first apply if needed the $\overset{c}{\vdash\!\dashv}$ step, then reduce all cuts above these $?_c$-rules (resp. $?_w$-rules) in the same way in both π_1 and π_2 before applying these commutations—thanks to weak normalization of $\xrightarrow{\overline{\beta}}$ (Corollary 13). This yields $\pi_1 \overset{c=}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}^*} \cdot \overset{r\setminus\top^*}{\vdash\!\dashv} \cdot \xleftarrow{\overline{\beta}^*} \pi_2$.

From now on, we assume both steps involve (at least) one common rule but do not share all of their rules. Observe this implies the two steps share exactly their unique *cut*-rule. Indeed, any other shared rule r is a non-*cut*-rule (remember we do not have $cut - cut$ commutations), meaning r is above a *cut*-rule that must also be shared between the two steps. Then, a simple case analysis proves that if two rules of a $\xrightarrow{\overline{\beta}}$ step are shared then all rules are.

As a consequence, neither of the considered $\xrightarrow{\overline{\beta}}$ step can be a step involving a *cut*-rule with the two rules immediately above it, *i.e.* no step can be a key case other than an *ax* one, and no step can be a $! - cut$ commutative step. We distinguish cases according to the kinds of these $\xrightarrow{\overline{\beta}}$ steps, exhausitivity following from the previous remark:

1. Both steps are *ax* key cases.

2. One step is an *ax* key case and the other a commutative case other than $! - cut$.

3. Both steps are commutative cases other than $! - cut$ ones.

1. If both steps are ax key cases. Here, we have a shared *cut*-rule with as premises two *ax*-rules. This critical pair leads to the same derivation from both choices of cut-elimination: $\pi_1 = \pi_2$.

2. If one step is an ax key case and the other a commutative case other than $! - cut$. By symmetry, assume $\pi \xrightarrow{\overline{\beta}} \pi_2$ is the *ax* key case. Considering all cases, we observe this *ax* key step can still be applied after the commutation to yield π_2, so that $\pi_1 \xrightarrow{\overline{\beta}^*} \pi_2$:

- $\pi_1 \xrightarrow{\overline{\beta}} \pi_2$ when $\pi \xrightarrow{\overline{\beta}} \pi_1$ is not a $\& - cut$, a $\cup - cut$, a $\top - cut$ nor a $\emptyset - cut$ commutative case;
- $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \pi_2$ when $\pi \xrightarrow{\overline{\beta}} \pi_1$ is a $\& - cut$ or a $\cup - cut$ commutative case;
- $\pi_1 = \pi_2$ when $\pi \xrightarrow{\overline{\beta}} \pi_1$ is a $\top - cut$ or a $\emptyset - cut$ commutative case.

3. If both steps are commutative cases other than $! - cut$ ones. As the reductions do not share both of their rules, in $\pi \xrightarrow{\overline{\beta}} \pi_1$ we sent a rule r_1 from a branch of the *cut* below it, and in $\pi \xrightarrow{\overline{\beta}} \pi_2$ we do similarly on a rule r_2 in the other branch. This case, more complex than the previous ones, is depicted schematically on Figure 6. We can in π_1 commute the *cut*-rule and r_2—doing it twice if r_1 is a &- or ∪-rule, and zero time if r_1 is a ⊤- or ∅-rule—obtaining π_1^1. We proceed similarly in π_2 by commuting the *cut*-rule and r_1, yielding π_2^1. The two resulting derivations differ exactly by a commutation of r_1 and r_2—except if both are ∅-rules, in which case $\pi_1^1 = \pi_2^1$ and we are done. For we apply rule commutations only on cut-free sub-derivations, we first eliminate all *cut*-rules above these two rules, in the same way in all sub-derivations (and in case of duplication, in the same way in all duplicates of the sub-derivations). This can be done thanks to weak normalization of $\xrightarrow{\overline{\beta}}$ (Corollary 13). Finally, we commute r_1 and r_2, yielding $\pi_1 \xrightarrow{\overline{\beta}^*} \pi_1^1 \xrightarrow{\overline{\beta}^*} \cdot \overset{r^=}{\vdash\!\dashv} \cdot \xleftarrow{\overline{\beta}^*} \pi_2^1 \xleftarrow{\overline{\beta}^*} \pi_2$ when both steps are commutative cases.

The only difficulty here is proving that the two derivations π_1^1 and π_2^1 differ by a commutation of r_1 and r_2 as claimed. This is a simple but tedious case analysis on the kind of rules r_1 and r_2 can be, and checking that in every case we indeed can apply an $\overset{r}{\vdash\!\dashv}$ step. As there are 17 possible commutative cases for $\pi \xrightarrow{\overline{\beta}} \pi_1$ and as many for $\pi \xrightarrow{\overline{\beta}} \pi_2$, this leads to $17^2 = 289$ cases. As such, we detail here only the case where r_1 is a &-rule and r_2 is a ⅋-rule. Here, our derivations are:

$$\pi = \underset{\rho_4}{\dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B, \Gamma} \qquad \overset{\rho_2}{\vdash A^\perp, C, \Gamma}}{\vdash A^\perp, B \,\&\, C, \Gamma}\,(\&) \qquad \dfrac{\overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash A, D \parr E, \Delta}\,(\parr)}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(cut)$$

$$\pi_1 = \underset{\rho_4}{\dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B, \Gamma} \qquad \dfrac{\overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash A, D \parr E, \Delta}\,(\parr)}{\vdash B, D \parr E, \Gamma, \Delta}\,(cut) \qquad \dfrac{\overset{\rho_2}{\vdash A^\perp, C, \Gamma} \qquad \dfrac{\overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash A, D \parr E, \Delta}\,(\parr)}{\vdash C, D \parr E, \Gamma, \Delta}\,(cut)}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(\&)$$

$$\pi_2 = \dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho_2}{\vdash A^\perp, C, \Gamma}}{\vdash A^\perp, B \,\&\, C, \Gamma}\,(\&) \qquad \overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash B \,\&\, C, D, E, \Gamma, \Delta}\,(cut)}{\underset{\rho_4}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(\parr)$$

Following the method described above, we apply two $\parr - cut$ commutative cases in π_1 and one $\& - cut$ commutative case in π_2, obtaining:

$$\pi_1^1 = \dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash B, D, E, \Gamma, \Delta}\,(cut)}{\vdash B, D \parr E, \Gamma, \Delta}\,(\parr) \qquad \dfrac{\dfrac{\overset{\rho_2}{\vdash A^\perp, C, \Gamma} \quad \overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash C, D, E, \Gamma, \Delta}\,(cut)}{\vdash C, D \parr E, \Gamma, \Delta}\,(\parr)}{\underset{\rho_4}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(\&)$$

$$\pi_2^1 = \dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash B, D, E, \Gamma, \Delta}\,(cut) \qquad \dfrac{\overset{\rho_2}{\vdash A^\perp, C, \Gamma} \quad \overset{\rho_3}{\vdash A, D, E, \Delta}}{\vdash C, D, E, \Gamma, \Delta}\,(cut)}{\vdash B \,\&\, C, D, E, \Gamma, \Delta}\,(\&)}{\underset{\rho_4}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(\parr)$$

Then, we eliminate all cut-rules above these &- and $\parr$-rules, in the same way in both derivations (thanks to Corollary 13), yielding:

$$\pi_1^2 = \dfrac{\dfrac{\overset{\tau_1^{cut-free}}{\vdash B, D, E, \Gamma, \Delta}}{\vdash B, D \parr E, \Gamma, \Delta}\,(\parr) \qquad \dfrac{\overset{\tau_2^{cut-free}}{\vdash C, D, E, \Gamma, \Delta}}{\vdash C, D \parr E, \Gamma, \Delta}\,(\parr)}{\underset{\rho_4}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(\&)$$

$$\pi_2^2 = \dfrac{\dfrac{\overset{\tau_1^{cut-free}}{\vdash B, D, E, \Gamma, \Delta} \qquad \overset{\tau_2^{cut-free}}{\vdash C, D, E, \Gamma, \Delta}}{\vdash B \,\&\, C, D, E, \Gamma, \Delta}\,(\&)}{\underset{\rho_4}{\vdash B \,\&\, C, D \parr E, \Gamma, \Delta}}\,(\parr)$$

We observe these two last derivations are equal up to a $\& - \parr$ commutation. Thus $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \pi_1^1 \xrightarrow{\overline{\beta}^+} \pi_1^2 \overset{r}{\vdash\!\dashv} \pi_2^2 \xleftarrow{\overline{\beta}^+} \pi_2^1 \xleftarrow{\overline{\beta}} \pi_2$. $\square$

4.2 Generalized local coherence for rule commutation

We prove here our instanciation of (Proposition 11).(iii), when considering a rule commutation.

Lemma 15. *Let π_1, π_2 and π_3 be derivations such that $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$. Then $\pi_1 \xrightarrow{\overline{\beta}^+} \cdot \xrightarrow{\top^*} \cdot \overset{r \setminus \top^*}{\vdash\!\dashv} \cdot \xleftarrow{\top^*} \cdot \xleftarrow{\overline{\beta}^+} \pi_2$. Diagrammatically:*

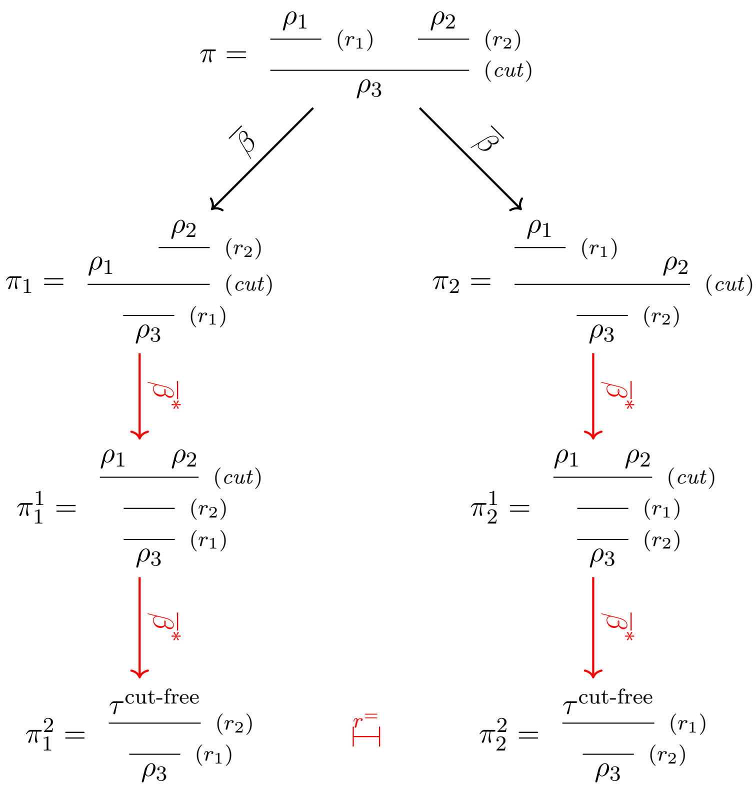
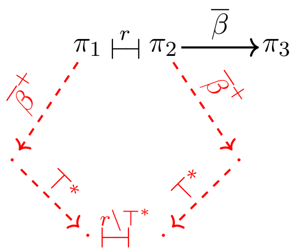


Figure 6: Schematic representation of case *3* in the proof of Lemma 14



Proof. Assume first that the $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$ and $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ steps involve distinct rules. Then, taking into account that rules of $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$ may be duplicated or erased by the $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ step, they commute and we have $\pi_1 \xrightarrow{\overline{\beta}} \cdot \left(\xrightarrow{\top^*} \cup \xleftarrow{\top^*} \cup \overset{r\setminus\top^*}{\vdash\!\dashv}\right) \pi_3 \xleftarrow{\overline{\beta}} \pi_2$. With more details:

- $\pi_1 \xrightarrow{\overline{\beta}} \cdot \overset{r}{\vdash\!\dashv} \pi_3$ if neither step duplicates nor erases the other;
- $\pi_1 \xrightarrow{\overline{\beta}} \cdot \overset{r}{\vdash\!\dashv} \cdot \overset{r}{\vdash\!\dashv} \pi_3$ if $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ duplicates a sub-derivation containing the rules involved in $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$;
- $\pi_1 \xrightarrow{\overline{\beta}} \pi_3$ if $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ erases a sub-derivation containing the rules involved in $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$.

There is no other case, as the $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$ step cannot duplicate nor erase the rules involved in $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ since rule commutations are applied on cut-free sub-derivations.

From now on, we suppose both steps involve at least one common rule, which in particular cannot be a *cut*-rule as no rule commutation in $\overset{r}{\vdash\!\dashv}$ involves such a rule. Observe there is exactly one shared rule because at most one rule involved in a $\overset{r}{\vdash\!\dashv}$ step can be above a *cut*-rule, and rules of a $\xrightarrow{\overline{\beta}}$ step are either *cut*-rules or rules above a *cut*-rule. We distinguish cases according to the kind of $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$:

1. $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ is a key case.

2. $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ is a commutative case.

1. If $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ *is a key case.* First, remark it cannot be an *ax* key case, because an *ax*-rule never commutes so the two steps share no rule, contradiction. We only treat here the $?_w - !$ key case. The cases where $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ is a $⅋ - \otimes$, $\& - \oplus_i$, $\perp - 1$, $?_d - !$, $?_w - !$ or $\forall - \exists$ key case are similar, and even simpler as less *cut*-rules (or exponential rules) result from the reduction.

By assumption, $\pi_1 \overset{r}{\mapsto} \pi_2$ was a step pushing down the $?_c$-rule (as the !-rule cannot commute), and up some non *cut*-rule r. Hence, the considered derivations are:

$$\pi_1 = \dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash ?A^\perp, ?A^\perp, \Gamma'}}{\vdash ?A^\perp, \Gamma'}\,(?_c)}{\vdash ?A^\perp, \Gamma}\,(r) \qquad \dfrac{\overset{\rho_2}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\underset{\rho_3}{\vdash \Gamma, ?\Delta}}\,(cut)$$

$$\pi_2 = \dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash ?A^\perp, ?A^\perp, \Gamma'}}{\vdash ?A^\perp, ?A^\perp, \Gamma}\,(r)}{\vdash ?A^\perp, \Gamma}\,(?_c) \qquad \dfrac{\overset{\rho_2}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\underset{\rho_3}{\vdash \Gamma, ?\Delta}}\,(cut)$$

$$\pi_3 = \dfrac{\dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash ?A^\perp, ?A^\perp, \Gamma'}}{\vdash ?A^\perp, ?A^\perp, \Gamma}\,(r) \qquad \dfrac{\overset{\rho_2}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash ?A^\perp, \Gamma, ?\Delta}\,(cut) \qquad \dfrac{\overset{\rho_2}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta, ?\Delta}\,(cut)}{\underset{\rho_3}{\vdash \Gamma, ?\Delta}}\,(?_c^*)$$

(up to symmetry, the case where the !-rule is on the left branch of the *cut*-rule and the $?_c$-rule on the right being solved similarly).

On Figure 7 is depicted the reasoning we apply here, along with the derivations we consider. We can in π_1 commute the *cut*-rule up and r down (as the main formula of r, if any, cannot be the formula on which we cut, and r is not an *ax*-rule), yielding $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1$ where:

$$\pi_1^1 := \dfrac{\dfrac{\dfrac{\overset{\rho_1}{\vdash ?A^\perp, ?A^\perp, \Gamma'}}{\vdash ?A^\perp, \Gamma'}\,(?_c) \qquad \dfrac{\overset{\rho_2}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma', ?\Delta}\,(cut)}{\underset{\rho_3}{\vdash \Gamma, ?\Delta}}\,(r)$$

Then, $\pi_1^1 \xrightarrow{\overline{\beta}} \pi_1^2$ using the same step as in $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$; unless r is a $\top$- or $\emptyset$-rule, in which case there is nothing to do and we set $\pi_1^2 := \pi_1^1$; and unless r is a &- or $\cup$-rule, where we have to apply this step in both occurrences, obtaining $\pi_1^1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \pi_1^2$. In any case, $\pi_1^1 \xrightarrow{\overline{\beta}^*} \pi_1^2$ with:

$$\pi_1^2 := \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\rho_1\\ \vdash ?A^\perp, ?A^\perp, \Gamma'\end{matrix} \qquad \dfrac{\begin{matrix}\rho_2\\ \vdash A, ?\Delta\end{matrix}}{\vdash !A, ?\Delta}\,(!)}{\vdash ?A^\perp, \Gamma', ?\Delta}\,(cut) \qquad \dfrac{\begin{matrix}\rho_2\\ \vdash A, ?\Delta\end{matrix}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma', ?\Delta, ?\Delta}\,(cut)}{\vdash \Gamma', ?\Delta}\,(?_c)}{\begin{matrix}\vdash \Gamma, ?\Delta\\ \rho_3\end{matrix}}\,(r)$$

Observe that π_1^2 is the derivation π_3 except that r is above some *cut*- and $?_c$-rules in π_3 and below them in π_1^2. Using $\xrightarrow{\overline{\beta}}$ in π_3, r can commute down the *cut*-rules created by the key case, yielding a derivation equal to π_1^2 up to rule commutations—as usual, we need first to eliminate all *cut*-rules above r in the same way in all sub-derivations (thanks to Corollary 13). Indeed, r has a main formula (if any) in Γ, hence not in $?\Delta$, and cannot be a !-rule. Therefore, $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \xrightarrow{\overline{\beta}^*} \pi_1^2 \xrightarrow{\overline{\beta}^*} \cdot \,(\xleftarrow{\top^*} \cup \overset{r\setminus\top^*}{\vdash\!\dashv}) \cdot \xleftarrow{\overline{\beta}^+} \pi_3 \xleftarrow{\overline{\beta}} \pi_2$, concluding this case. For the other key cases, we also obtain $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \xrightarrow{\overline{\beta}^*} \pi_1^2 \xrightarrow{\overline{\beta}^*} \cdot \,(\xleftarrow{\top^*} \cup \overset{r\setminus\top^*}{\vdash\!\dashv}) \cdot \xleftarrow{\overline{\beta}^*} \pi_3 \xleftarrow{\overline{\beta}} \pi_2$, with the $\xleftarrow{\top^*} \cup \overset{r\setminus\top^*}{\vdash\!\dashv}$ step also needed for r to commute with the $?_w$-rules produced by a $?_w - !$ key case (the other key cases need no such steps).

2. If $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ *is a commutative case.* As $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$ and $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ have exactly one rule in common, the $\overset{r}{\vdash\!\dashv}$ step involves the rule r that will be commuted down in the $\xrightarrow{\overline{\beta}}$ step, and another rule that we call s (r and s are not *cut*-rules). The derivation π_1 has from top to bottom r, s and *cut*, π_2 has s, r and *cut*, and π_3 has s, *cut* and r. We will also consider the rule t on the other branch of the *cut*-rule. Schematically (and up to symmetry):

$$\pi_1 = \dfrac{\dfrac{\dfrac{}{\quad\;}\,(r)}{\quad\;}\,(s) \qquad \dfrac{}{\quad\;}\,(t)}{\qquad\qquad}\,(cut) \qquad \pi_2 = \dfrac{\dfrac{\dfrac{}{\quad\;}\,(s)}{\quad\;}\,(r) \qquad \dfrac{}{\quad\;}\,(t)}{\qquad\qquad}\,(cut) \qquad \pi_3 = \dfrac{\dfrac{\dfrac{}{\quad\;}\,(s) \qquad \dfrac{}{\quad\;}\,(t)}{\qquad\qquad}\,(cut)}{\quad\;}\,(r)$$

We have here different sub-cases, according to whether the *cut*-rule commutes with s and/or t. More exactly, we consider the following exhaustive cases:

2.a. s commutes with the *cut*-rule;

2.b. t commutes with the *cut*-rule;

2.c. t is an *ax*-rule;

2.d. neither s nor t commute with the *cut*-rule and t is not an *ax*-rule.

2.a. If s commutes with the cut-rule. Our reasoning for this case is depicted on Figure 8. In π_1, we commute the *cut*-rule successively with s and r, yielding $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \xrightarrow{\overline{\beta}^*} \pi_1^2$ (the $\xrightarrow{\overline{\beta}^*}$ being of length one, except if s is a &- or $\oplus$-rule, in which case we apply the commutation with r for both occurrences, or if s is a $\top$- or $\emptyset$-rule, in which case there is no commutation to apply). The derivation π_1^2 has from top to bottom *cut*, r and s. Meanwhile, in π_3 we commute the *cut*-rule with s (twice if r is a &- or $\oplus$-rule, or zero time if r is a $\top$- or $\emptyset$-rule), yielding π_3^1 having from top to bottom *cut*, s and r. Now, both π_1^2 and π_3^1 have above r and s a same derivation (maybe duplicated or erased). We use normalization of $\xrightarrow{\overline{\beta}}$ (Corollary 13) to eliminate all *cut*-rules in this sub-derivation, in the same way for all its occurrences in π_1^2 and π_3^1, obtaining derivations π_1^3 and π_3^2 equal up to the commutation of r and s: the very same one that was used in $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$. We thus obtain $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \xrightarrow{\overline{\beta}^*} \pi_1^2 \xrightarrow{\overline{\beta}^*} \pi_1^3 \overset{r}{\vdash\!\dashv} \pi_3^2 \xleftarrow{\overline{\beta}^*} \pi_3^1 \xleftarrow{\overline{\beta}^*} \pi_3 \xleftarrow{\overline{\beta}} \pi_2$. To formally prove that we indeed can apply a $\overset{r}{\vdash\!\dashv}$ step, we would have to check all cases depending on the kind of rules

r and s are. We do not write this tedious case study; a motivated reader can easily check some of these cases.

2.b. If t commutes with the cut-rule. This case is represented on Figure 9. On π_1, we apply a commutative $\xrightarrow{\overline{\beta}}$ step between t and the *cut*-rule, giving a derivation π_1^1. We apply a similar step on π_2, obtaining π_2^1. Observe that π_1^1 and π_2^1 differ by the same rule commutation between r and s that was done in $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$ (as usual, by twice this commutation if t is a &- or $\cup$-rule and they are equal if t is a $\top$- or $\emptyset$-rule). Thence $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \ (\xrightarrow{\top^*} \cup \xleftarrow{\top^*} \cup \overset{r\setminus\top^*}{\vdash\!\dashv}) \ \pi_2^1 \xleftarrow{\overline{\beta}} \pi_2$.

2.c. If t is an ax-rule. This case is represented on Figure 10. In both π_1 and π_3, we apply an *ax* key case using t, giving respectively π_1^1 and π_3^1 (as usual, in π_3 we do this key case twice if r is a &- or $\cup$-rule and have nothing to do if r is a $\top$- or $\emptyset$-rule). Then $\pi_1^1 \overset{r}{\vdash\!\dashv} \pi_3^1$ through a commutation between r and s, using the very same rule commutation as in $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$—and with no *cut*-rule above r and s as there was a rule commutation $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2$. Thus $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \overset{r}{\vdash\!\dashv} \pi_3^1 \xleftarrow{\overline{\beta}^*} \pi_3 \xleftarrow{\overline{\beta}} \pi_2$.

2.d. If s and t both do not commute with the cut-rule and t is not an ax-rule. This case is represented on Figure 11. Observe here that a key case can be applied using the *cut*-rule, s and t. The main point of attention here is the case where t is a !-rule: we have to prove its main formula is the formula that is cut (this is the only rule that may not commute with a *cut*-rule not on its principal formula, and neither s nor r can be a !-rule as they commute). If it were not the case, then the cut formula on the branch of r and s must be a !-formula. As neither r nor s are !-rules, s must commute with the *cut*-rule which contradicts our hypothesis.

Hence, we apply a key case using s and t in both π_1 and π_3, obtaining respectively derivations π_1^1 and π_3^1. We observe that π_3^1 can be obtained from π_1^1 by commuting r with the rules produced by this key case, that is 0, 1 or 2 *cut*-rules and possibly some $?_c$- or $?_w$-rules below these *cut*-rules. Thus $\pi_1 \xrightarrow{\overline{\beta}} \pi_1^1 \xrightarrow{\overline{\beta}^*} \cdot \ (\overset{r\setminus\top^*}{\vdash\!\dashv} \cup \xrightarrow{\top^*}) \ \pi_3^1 \xleftarrow{\overline{\beta}^*} \pi_3 \xleftarrow{\overline{\beta}} \pi_2$. ☐

4.3 Generalized local coherence for the cut-cut commutative step

We prove here our instanciation of (Proposition 11).(iii), when considering a cut-cut commutative step. To this end, we need an intermediate result about commutativity of the *cut*-rule.

4.3.1 Commutativity of the *cut*-rule

The following intermediate result will be needed: two derivations that are the same up to commutativity of a *cut*-rule, reduce to a same derivation.

Definition 16. The **commutativity of a *cut*-rule** is the symmetric relation $\overset{sc}{\vdash\!\dashv}$ on derivations defined by:

$$\pi = \dfrac{\overset{\phi_1}{\vdash A, \Gamma} \qquad \overset{\phi_2}{\vdash A^\perp, \Delta}}{\underset{\phi_3}{\vdash \Gamma, \Delta}}\ (cut) \quad \overset{sc}{\vdash\!\dashv} \quad \tau = \dfrac{\overset{\phi_2}{\vdash A^\perp, \Delta} \qquad \overset{\phi_1}{\vdash A, \Gamma}}{\underset{\phi_3}{\vdash \Gamma, \Delta}}\ (cut)$$

This relation $\overset{sc}{\vdash\!\dashv}$ is included in $=_\beta$, using a $cut - cut$ commutation. Indeed, consider the two following derivations:

$$\pi' := \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\ (ax) \qquad \overset{\phi_1}{\vdash A, \Gamma}}{\vdash A, \Gamma}\ (cut) \qquad \overset{\phi_2}{\vdash A^\perp, \Delta}}{\underset{\phi_3}{\vdash \Gamma, \Delta}}\ (cut) \quad \text{and} \quad \tau' := \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\ (ax) \qquad \overset{\phi_2}{\vdash A^\perp, \Delta}}{\vdash A^\perp, \Delta}\ (cut) \qquad \overset{\phi_1}{\vdash A, \Gamma}}{\underset{\phi_3}{\vdash \Gamma, \Delta}}\ (cut)$$

Then $\pi \xleftarrow{\overline{\beta}} \pi' \overset{c}{\vdash\!\dashv} \tau' \xrightarrow{\overline{\beta}} \tau$, these $\xrightarrow{\overline{\beta}}$ steps being *ax* key cases. We need, and prove, a stronger result: $\overset{sc}{\vdash\!\dashv}$ is included in $\overline{\beta}$-equality $=_{\overline{\beta}}$.

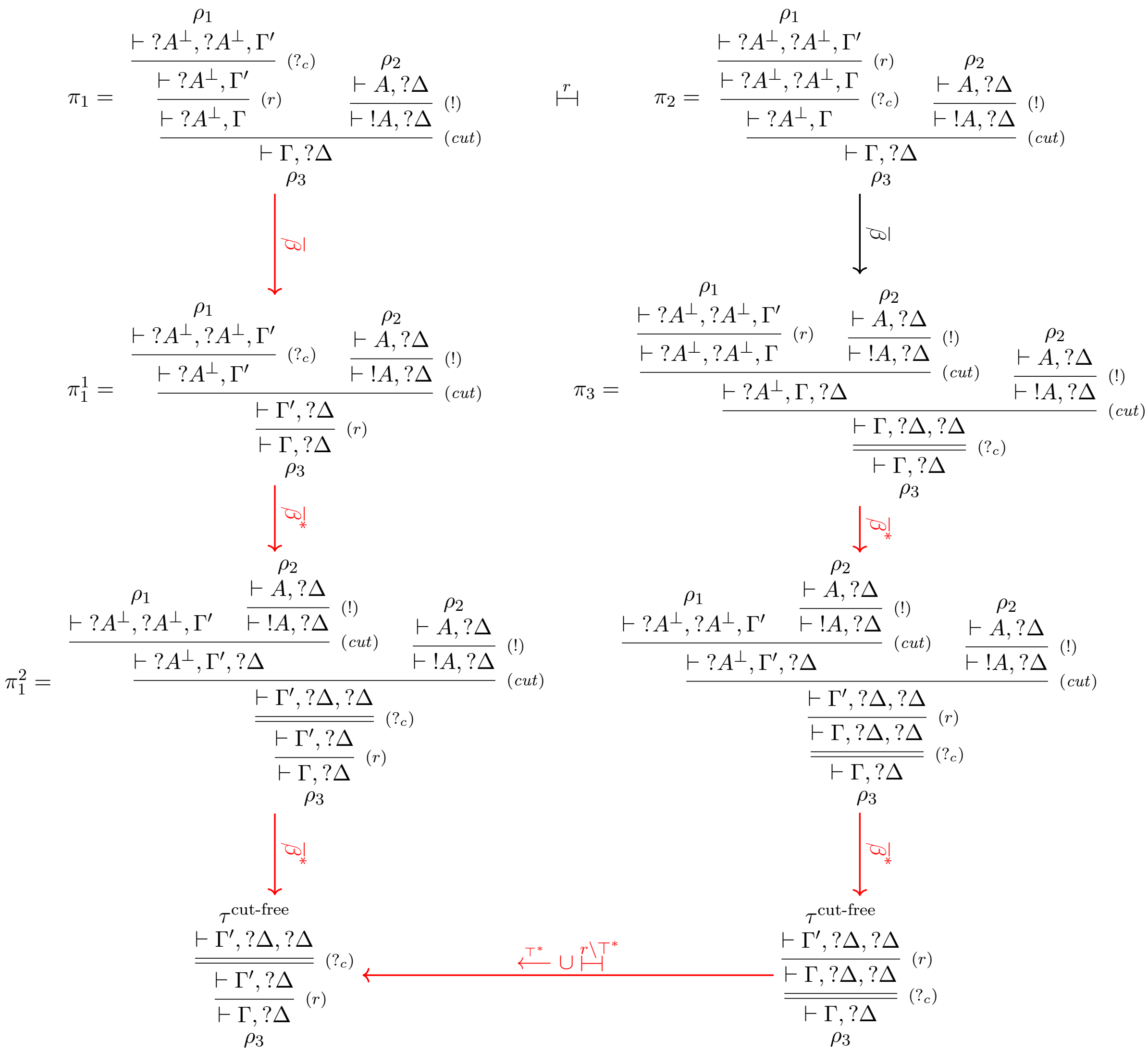


Figure 7: Representation of case *1* in the proof of Lemma 15

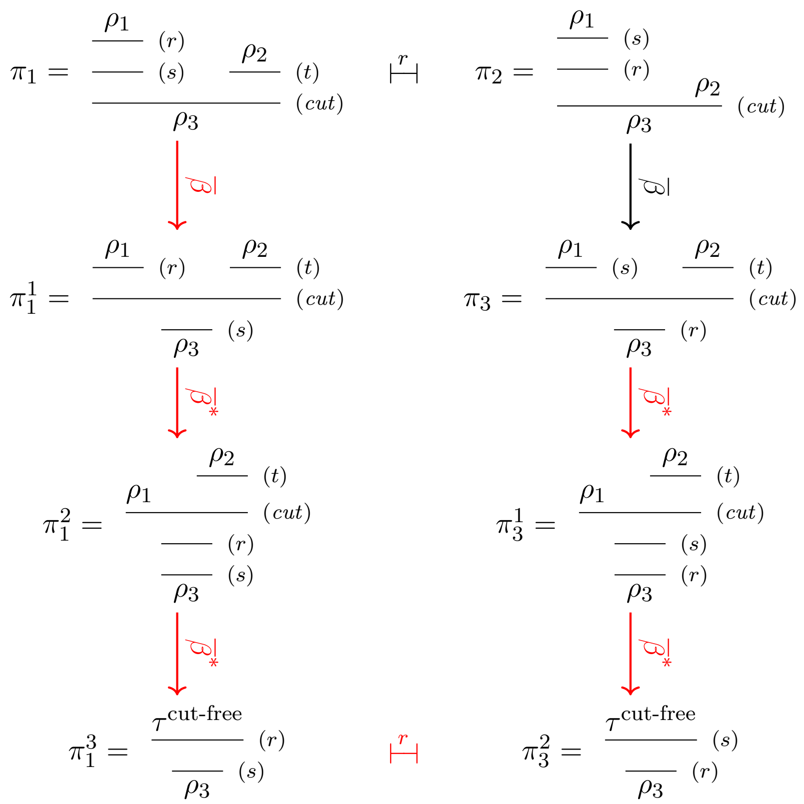


Figure 8: Schematic representation of case *2.a* in the proof of Lemma 15

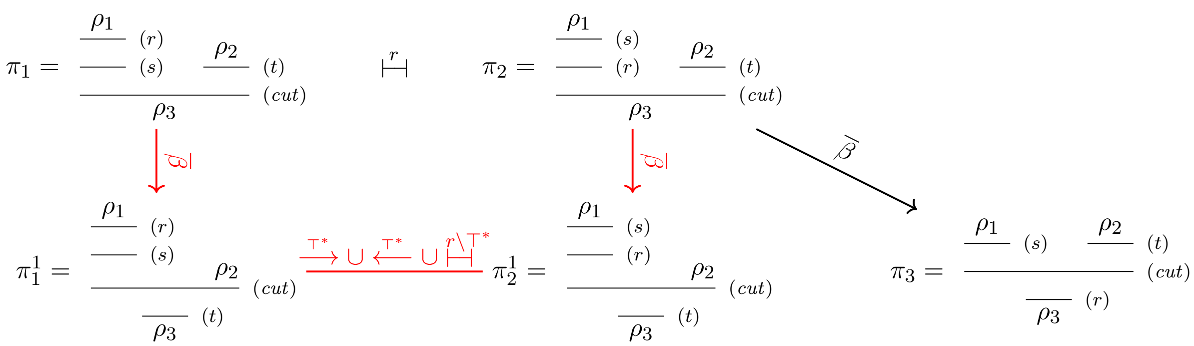


Figure 9: Schematic representation of case *2.b* in the proof of Lemma 15

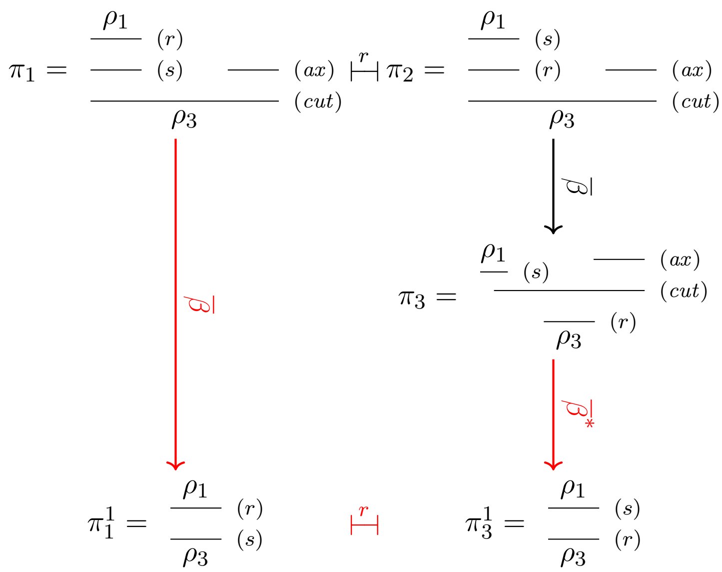


Figure 10: Schematic representation of case 2.c in the proof of Lemma 15

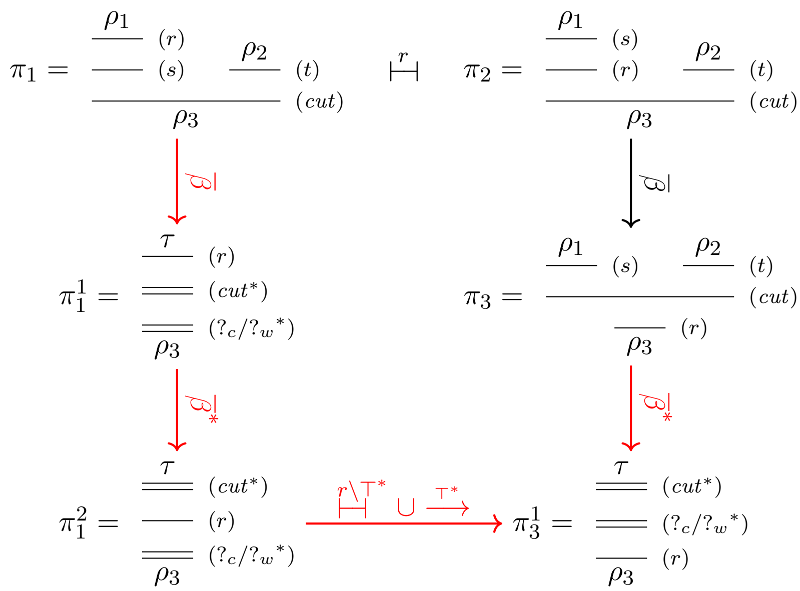


Figure 11: Schematic representation of case 2.d in the proof of Lemma 15

Lemma 17. *Let π and τ be two derivations equal up to symmetries of cut-rules: $\pi \overset{sc^*}{\vdash\!\dashv} \tau$. Then $\pi \xrightarrow{\overline{\beta}^*} \cdot \xleftarrow{\overline{\beta}^*} \tau$.*

Proof. The idea is to apply cut-elimination steps in π and τ in a symmetric way and to show the resulting derivations are related by $\overset{sc^*}{\vdash\!\dashv}$. The result then follows through weak normalization of cut-elimination, as two cut-free derivations equal up to $\overset{sc^*}{\vdash\!\dashv}$ are simply equal.

We reason by induction on a finite sequence $\pi \xrightarrow{\overline{\beta}^*} \pi'$ with π' some cut-free derivation found by weak normalization (Corollary 13). If this sequence is empty, then $\pi = \pi'$ is *cut*-free, hence $\pi = \tau$. Thus, consider the first step $\pi \xrightarrow{\overline{\beta}} \pi_1$ of this sequence, which involves a unique *cut*-rule c. We apply the corresponding step in τ: either c has the same premises in π and τ, or they are switched; in both cases, the same kind of $\xrightarrow{\overline{\beta}}$ step can be applied in τ. We obtain $\tau \xrightarrow{\overline{\beta}} \tau_1$ with in τ_1 possibly some *cut*-rules still symmeterized compared to π_1: those are the *cut*-rules that were with switched premises between π and τ, other than c and those erased by the $\xrightarrow{\overline{\beta}}$ steps; or the *cut*-rules duplicated by the $\xrightarrow{\overline{\beta}}$ step; or the *cut*-rules introduced by the $\xrightarrow{\overline{\beta}}$ step. Checking all possible cases for $\pi \xrightarrow{\overline{\beta}} \pi_1$, we always get $\pi_1 \overset{sc^*}{\vdash\!\dashv} \tau_1$. We conclude by induction hypothesis: $\pi \xrightarrow{\overline{\beta}} \pi_1 \xrightarrow{\overline{\beta}^*} \cdot \xleftarrow{\overline{\beta}^*} \tau_1 \xleftarrow{\overline{\beta}} \tau$. □

4.3.2 Cut-cut commutative steps and equality up to cut-elimination

Using this intermediate result, we now prove the wanted hypothesis.

Lemma 18. *Let π and τ be derivations such that $\pi \overset{c}{\vdash\!\dashv} \tau$. Then $\pi \xrightarrow{\overline{\beta}^+} \cdot \overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}^*} \cdot \xleftarrow{\overline{\beta}^*} \cdot \overset{c^*}{\vdash\!\dashv} \cdot \xleftarrow{\overline{\beta}^+} \tau$. Diagrammatically:*

$$\begin{tikzcd}
 & \pi \arrow[rr, phantom, "\overset{c}{\vdash\!\dashv}"] \arrow[d, dashed, "\overline{\beta}^{+}"] & & \tau \arrow[d, dashed, "\overline{\beta}^{+}"] & \\
\cdot \arrow[r, phantom, "\overset{c^*}{\vdash\!\dashv}"] \arrow[drr, dashed, "\overline{\beta}^{*}"'] & \cdot & & \cdot \arrow[r, phantom, "\overset{c^*}{\vdash\!\dashv}"] & \cdot \arrow[dll, dashed, "\overline{\beta}^{*}"] \\
 & & \cdot & &
\end{tikzcd}$$

Proof. As $\pi \overset{c}{\vdash\!\dashv} \tau$, we have by definition:

$$\pi = \dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B^\perp, \Gamma_1} \quad \overset{\rho_2}{\vdash B, \Gamma_2}}{\vdash A^\perp, \Gamma_1, \Gamma_2}\,(cut) \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,(cut) \qquad \tau = \dfrac{\dfrac{\overset{\rho_1}{\vdash A^\perp, B^\perp, \Gamma_1} \quad \overset{\rho_3}{\vdash A, \Gamma_3}}{\vdash B^\perp, \Gamma_1, \Gamma_3}\,(cut) \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,(cut)$$

(or one of the three other analogous situations, up to switching branches of the two *cut*-rules). By eliminating cuts in ρ_1, ρ_2 and ρ_3 using $\xrightarrow{\overline{\beta}}$ (Corollary 13), we can assume them to be cut-free. We proceed through a case study of the last rules of ρ_1, ρ_2 and ρ_3, that we call respectively r_1, r_2 and r_3. Observe that r_1 cannot be a 1-rule, for its conclusion sequent has several formulas (at least $A^\perp$ and $B^\perp$), and that none of the r_i can be a mix_0-rule. By symmetry, we can assume that the main formula of r_1 (if any) is not $A^\perp$—otherwise switch π and τ, hence $A^\perp$ and $B^\perp$. Our cases are the followings:

1. One of the r_i is an *ax*-rule.

2. The rule r_1 is not a !-rule nor a rule whose main formula is $B^\perp$; or r_2 is not a !-rule nor a rule whose main formula is B; or r_3 is not a !-rule nor a rule whose main formula is A.

3. The rule r_1 is a !-rule whose main formula is not $B^\perp$.

4. The rule r_2 is a !-rule whose main formula is not B.

5. The rule r_1 has for main formula $B^\perp$ and r_2 has for main formula B.

This list of cases is exhaustive. Indeed, if r_1 (resp. r_2) is not a rule whose main formula is $B^\perp$ (resp. B), then we are in case *1*, *2*, *3* or *4*; otherwise, we are in case *5*. We consider the rule r_3 in cases *1* and *2* in order to have more constrained cases *3* to *5*. In particular, we have the following property:

$$\text{If } r_1 \text{ is a !-rule, then either we are in case } \mathit{2} \text{ or } r_3 \text{ is a !-rule of main formula } A. \tag{$*$}$$

Let us prove this claimed ($*$). There, $A^\perp$ is a ?-formula since it cannot be the main formula of r_1, so A is a !-formula. Thence, either r_3 is a !-rule introducing A as wanted, or r_3 is not a !-rule and has not A as a main formula—which is an instance of case *2* of this proof. This concludes the demonstration of ($*$), that we will use in cases *3* to *5*.

1. If one of the r_i is an ax-rule. If r_2 or r_3 is an ax-rule, then applying an ax key case with it in π yields the same derivation as applying this step in τ: $\pi \xrightarrow{\overline{\beta}} \cdot \xleftarrow{\overline{\beta}} \tau$. If r_1 is an ax-rule, applying an ax key case with it in π and τ yields two derivations equal up to commutativity of a cut-rule, so by Lemma 17 we obtain $\pi \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}^*} \cdot \xleftarrow{\overline{\beta}^*} \cdot \xleftarrow{\overline{\beta}} \tau$. Thus, we assume from now on that no r_i is an ax-rule.

2. If r_1 is not a !-rule nor a rule whose main formula is $B^\perp$; or r_2 is not a !-rule nor a rule whose main formula is B; or r_3 is not a !-rule nor a rule whose main formula is A. We consider only the first possibility, represented on Figure 12. The other two are similar and simpler. We have that r_1 is not a rule whose main formula is $B^\perp$, nor a rule whose main formula is $A^\perp$ (by hypothesis before the case study), nor a !-rule, nor an ax-rule (otherwise we are in case *1*), nor a 1-rule (it has $B^\perp$ in its conclusion sequent but it is not its main formula), nor a mix_0-rule, nor a cut-rule. Hence, looking at the kind of rule r_1 can be, we get that r_1 commutes with the cut-rule on B and with the cut-rule on A (r_1 can be a $\parr$- $\otimes$- $\bot$- $\&$- $\oplus_1$- $\oplus_2$- $\top$- $?_d$- $?_c$- $?_w$- $\forall$- $\exists$- mix_2- $\cup$- or $\emptyset$-rule). By applying in both π and τ two commutative cases on r_1 with the two successive cut-rules below it, we obtain respectively the derivations:

$$\pi_1 := \dfrac{\dfrac{\dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1} \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\vdash A^\perp, \Gamma'_1, \Gamma_2}\,(cut) \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\vdash \Gamma'_1, \Gamma_2, \Gamma_3}\,(cut)}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,(r_1) \qquad \tau_1 := \dfrac{\dfrac{\dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1} \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\vdash B^\perp, \Gamma'_1, \Gamma_3}\,(cut) \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\vdash \Gamma'_1, \Gamma_2, \Gamma_3}\,(cut)}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,(r_1)$$

with $\rho_1 := \dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1}}{\vdash A^\perp, B^\perp, \Gamma_1}\,(r_1)$ (with our usual abuse of notations, *e.g.* r_1 may be a binary rule), $\pi \xrightarrow{\overline{\beta}^+} \pi_1$ and $\tau \xrightarrow{\overline{\beta}^+} \tau_1$. In each case, we observe $\pi_1 \overset{c^*}{\vdash\!\dashv} \tau_1$ by means of two such steps in case r_1 is a $\&$- or $\cup$-rule, by means of zero step if r_1 is a $\top$- or $\emptyset$-rule, and by means of one step otherwise.

3. If r_1 is a !-rule whose main formula is not $B^\perp$

This case is represented on Figure 13. Call $!C$ the main formula of r_1, and remember it cannot be $A^\perp$. As $A^\perp$ and $B^\perp$ belong to the conclusion sequent of r_1, they must be ?-formulas, thus A and B are !-formulas. Thanks to ($*$), r_3 is a !-rule on A. Similarly, r_2 is a !-rule on B: if it were not the case, then r_2 is not a !-rule and its main formula (if any) is not B, thus we can go to case *2* of this proof.

Remark here that we can apply in π two successive $! - cut$ commutative steps on r_1, obtaining a derivation π_1, and similarly in τ, obtaining τ_1. We then observe that $\pi_1 \overset{c}{\mapsto\!\dashv} \tau_1$, see Figure 13. We conclude that $\pi \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \pi_1 \overset{c}{\mapsto\!\dashv} \tau_1 \xleftarrow{\overline{\beta}} \cdot \xleftarrow{\overline{\beta}} \tau$.

4. If r_2 is a !-rule whose main formula is not B.

Suppose that r_2 is a !-rule whose main formula $!C$ is not B. This case, more complex than the previous one, is depicted on Figure 14. Observe B is a ?-formula, meaning $B^\perp$ is a !-formula. Consider r_1: either it is a !-rule on $B^\perp$, or it is not a !-rule and its main formula (if any) is not $B^\perp$. If it is the later, we go to case *2*, so suppose r_1 is a !-rule on $B^\perp$. By ($*$), r_3 can only be a !-rule on A (or we are in case *2*). We proceed as follows—see Figure 14 for the relevant derivations. In τ, we apply a $! - cut$ commutative step on r_1 which then enables another $! - cut$ commutative step on r_2, yielding $\tau \xrightarrow{\overline{\beta}} \tau_1 \xrightarrow{\overline{\beta}} \tau_2$. Meanwhile, in π, we apply two successive $! - cut$ commutative steps on r_2, obtaining $\pi \xrightarrow{\overline{\beta}} \pi_1 \xrightarrow{\overline{\beta}} \pi_2$. We then observe that applying a $cut - cut$ commutative case in π_2 allows us to apply a $! - cut$ commutative step on r_1, that yields exactly τ_2. Hence, we have $\pi \xrightarrow{\overline{\beta}} \pi_1 \xrightarrow{\overline{\beta}} \pi_2 \overset{c}{\mapsto\!\dashv} \cdot \xrightarrow{\overline{\beta}} \tau_2 \xleftarrow{\overline{\beta}} \tau_1 \xleftarrow{\overline{\beta}} \tau$.

5. The rule r_1 has for main formula $B^\perp$ and r_2 has for main formula B.

Here, r_1 and r_2 make a key step in π. We claim that r_1 commutes with the uppermost cut-rule on A in τ. Indeed, as r_1 introduces $B^\perp$, the only case where it may not commute in τ is if it is a !-rule (for it cannot be an ax- nor a 1- nor a mix_0-rule). But then r_3 is a !-rule of main formula A thanks to ($*$), and one can apply a $! - cut$ commutative case with r_1 in τ.

Let us now explain how to proceed. We apply a key case in π, with the following ordering of $?_c$- and $?_w$-rules when this step is respectively a $?_c - !$ or $?_w - !$ key case: from top to bottom, we first put eventual rules on A and $A^\perp$, then on Γ_3, next on Γ_2, and finally on Γ_1 (there cannot be any on B nor $B^\perp$ for it is the main formula on which we cut). Meanwhile, in τ, we apply a commutative case on r_1 followed by the very same key case—in particular, for a $?_c - !$ or $?_w - !$ key case we choose to put the resulting $?_c$- or $?_w$-rules following the same strategy, and taking the same order when introducing rules on some Γ_i. The derivations we obtain are

$$\pi_1 := \cfrac{\cfrac{\cfrac{\cfrac{\rho'_1}{\vdash A^\perp, \Gamma'_1} \quad \cfrac{\rho'_2}{\vdash \Gamma'_2}}{\vdash A^\perp, \Gamma'_1, \Gamma'_2}\,(cut^*)}{\vdash A^\perp, \Gamma_1, \Gamma_2}\,(?_c/?_w{}^*) \qquad \cfrac{\rho_3}{\vdash A, \Gamma_3}}{\cfrac{\vdash \Gamma_1, \Gamma_2, \Gamma_3}{\rho_4}}\,(cut) \qquad \tau_1 := \cfrac{\cfrac{\cfrac{\cfrac{\rho'_1}{\vdash A^\perp, \Gamma'_1} \quad \cfrac{\rho_3}{\vdash A, \Gamma_3}}{\vdash \Gamma'_1, \Gamma_3}\,(cut) \qquad \cfrac{\rho'_2}{\vdash \Gamma'_2}}{\vdash \Gamma'_1, \Gamma'_2, \Gamma_3}\,(cut^*)}{\cfrac{\vdash \Gamma_1, \Gamma_2, \Gamma_3}{\rho_4}}\,(?_c/?_w{}^*)$$

with $\rho_1 := \cfrac{\cfrac{\rho'_1}{\vdash A^\perp, \Gamma'_1}}{\vdash A^\perp, B^\perp, \Gamma_1}\,(r_1)$, $\rho_2 := \cfrac{\cfrac{\rho'_2}{\vdash \Gamma'_2}}{\vdash B, \Gamma_2}\,(r_2)$, $\pi \xrightarrow{\overline{\beta}} \pi_1$ and $\tau \xrightarrow{\overline{\beta}^+} \tau_1$. The notation cut^* here means we have a number of cut-rules (0, 1 or 2), and $?_c/?_w{}^*$ some $?_c$- or $?_w$-rules that could be produced by a $?_c - !$ or $?_w - !$ key case.

If the key cases were not $?_c - !$ nor $?_w - !$, we observe that $\pi_1 \overset{c^*}{\mapsto\!\dashv} \tau_1$—this sequence being of size the number of cut-rules represented by cut^*—allowing us to conclude. This is represented on Figure 15. As usual, to be formal, we would need a case study on which key case is made by r_1 and r_2, checking for each instance that the above scheme holds: a motivated reader could easily do so.

Thus, assume to be in the $?_c - !$ or $?_w - !$ key steps. We first consider the easier situation where the !-rule is r_2, while r_1 is the $?_c$- or $?_w$-rule. In this case, the sequence of new $?_c$- or

$$\pi = \dfrac{\dfrac{\dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1}}{\vdash A^\perp, B^\perp, \Gamma_1}\,{\scriptstyle (r_1)} \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\vdash A^\perp, \Gamma_1, \Gamma_2}\,{\scriptstyle (cut)} \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,{\scriptstyle (cut)} \quad \overset{c}{\vdash\!\dashv} \quad \tau = \dfrac{\dfrac{\dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1}}{\vdash A^\perp, B^\perp, \Gamma_1}\,{\scriptstyle (r_1)} \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\vdash B^\perp, \Gamma_1, \Gamma_3}\,{\scriptstyle (cut)} \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,{\scriptstyle (cut)}$$

$$\big\downarrow \overline{\beta}^+ \qquad\qquad\qquad\qquad \big\downarrow \overline{\beta}^+$$

$$\pi_1 = \dfrac{\dfrac{\dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1} \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\vdash A^\perp, \Gamma'_1, \Gamma_2}\,{\scriptstyle (cut)} \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\vdash \Gamma'_1, \Gamma_2, \Gamma_3}\,{\scriptstyle (cut)}}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,{\scriptstyle (r_1)} \quad \overset{c^*}{\vdash\!\dashv} \quad \tau_1 = \dfrac{\dfrac{\dfrac{\overset{\rho'_1}{\vdash A^\perp, B^\perp, \Gamma'_1} \qquad \overset{\rho_3}{\vdash A, \Gamma_3}}{\vdash B^\perp, \Gamma'_1, \Gamma_3}\,{\scriptstyle (cut)} \qquad \overset{\rho_2}{\vdash B, \Gamma_2}}{\vdash \Gamma'_1, \Gamma_2, \Gamma_3}\,{\scriptstyle (cut)}}{\underset{\rho_4}{\vdash \Gamma_1, \Gamma_2, \Gamma_3}}\,{\scriptstyle (r_1)}$$

Figure 12: Representation of case *2* in the proof of Lemma 18

$?_w$-rules is on Γ_2 in both π_1 and τ_1, and in π_1 each of these rules can commute with the *cut*-rule on A. Applying these successive commutations in π_1 yields $\pi_1 \xrightarrow{\overline{\beta}^*} \pi_2$, where $\pi_2 \overset{c^*}{\vdash\!\dashv} \tau_1$ (with no need for any rule commutation): we are done.

Therefore, we suppose from now on that the !-rule is r_1, of main formula $B^\perp$, and that r_2 is the $?_c$- or $?_w$-rule. This implies r_3 is a !-rule of main formula A by $(*)$. We now distinguish two cases according to the kind of the key step, *i.e.* consider whether r_2 is a $?_c$- or a $?_w$-rule.

First, assume we applied a $?_c - !$ key step, *i.e.* that r_2 is a $?_c$-rule. To be more explicit, we denote $A = !A'$, $B = ?B'$, and indicate by $?\Gamma_1$ and $?\Gamma_3$ that all of these formulas are ?-formulas. All the derivations considered in this case are depicted on Figure 16. Remember that, starting from π and τ, we applied in π a $?_c - !$ key case and in τ a $! - !$ commutative then a $?_c - !$ key step, obtaining respectively π_1 and τ_1. Remark the $?_c$-rules on $?\Gamma_1$ in π_1 can be commuted with the *cut*-rule on $!A'$, giving $\pi_1 \xrightarrow{\overline{\beta}^*} \pi_2$. Now, a $?_c - !$ key case can be applied on the bottom *cut*-rule of π_2, giving a derivation π_3 (taking care that the introduced $?_c$-rules on $?\Gamma_3$ are introduced in the same order they appear in τ_1). We then apply a sequence of three $cut - cut$ commutative cases in π_3 to obtain a derivation π_4 such that applying two $! - !$ commutative cases in π_4 yields exactly τ_1. This concludes the case where r_2 is a $?_c$-rule: we have $\pi \xrightarrow{\overline{\beta}} \pi_1 \xrightarrow{\overline{\beta}^*} \pi_2 \xrightarrow{\overline{\beta}} \pi_3 \overset{c^*}{\vdash\!\dashv} \pi_4 \xrightarrow{\overline{\beta}^+} \tau_1 \xleftarrow{\overline{\beta}^+} \tau$.

The case where r_2 is a $?_w$-rule is similar and simpler. We get $\pi \xrightarrow{\overline{\beta}} \pi_1 \xrightarrow{\overline{\beta}^*} \pi_2 \xrightarrow{\overline{\beta}} \tau_1 \xleftarrow{\overline{\beta}} \cdot \xleftarrow{\overline{\beta}} \tau$ as follows—see Figure 17 for the derivations encountered. In π, apply the $?_w - !$ key step to obtain π_1, then commute down the new $?_w$-rules on $?\Gamma_1$ to get π_2, then observe a new $?_w - !$ key case can be applied thanks to the new $?_w$-rule on $A^\perp = ?A'^\perp$. Meanwhile, in τ, one can apply a $! - !$ commutative step followed by a $?_w - !$ key step to directly get the same resulting τ_1. This concludes this case and the proof. $\square$

Remark 19. Our proof of Lemma 18 is quite complex. A simpler proof would be by means of a property such as: if $\pi_1 \overset{c}{\vdash\!\dashv} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$, then $\pi_1 \xrightarrow{\overline{\beta}^*} \cdot \overset{c^*}{\vdash\!\dashv} \pi_3$. Unfortunately, this property does not hold in linear logic due to the !-rule: see [Di25] for a counter-example.

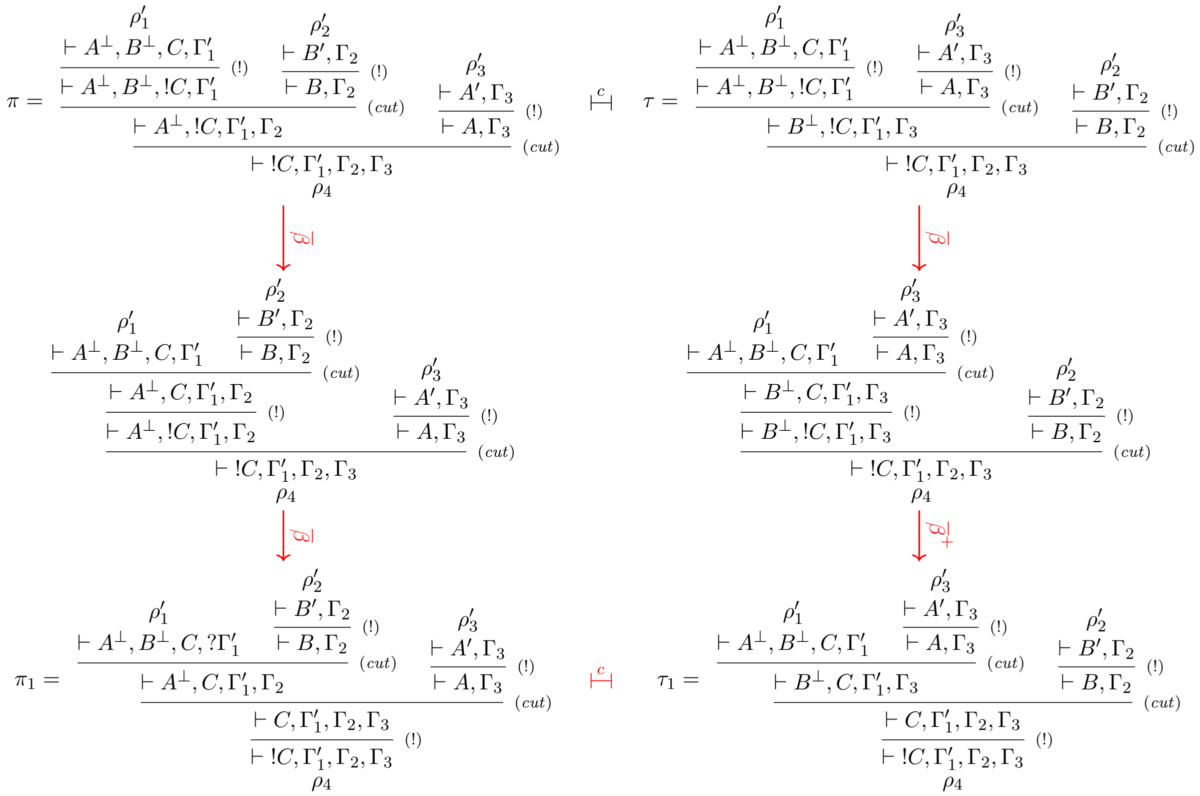


Figure 13: Representation of case *3* in the proof of Lemma 18

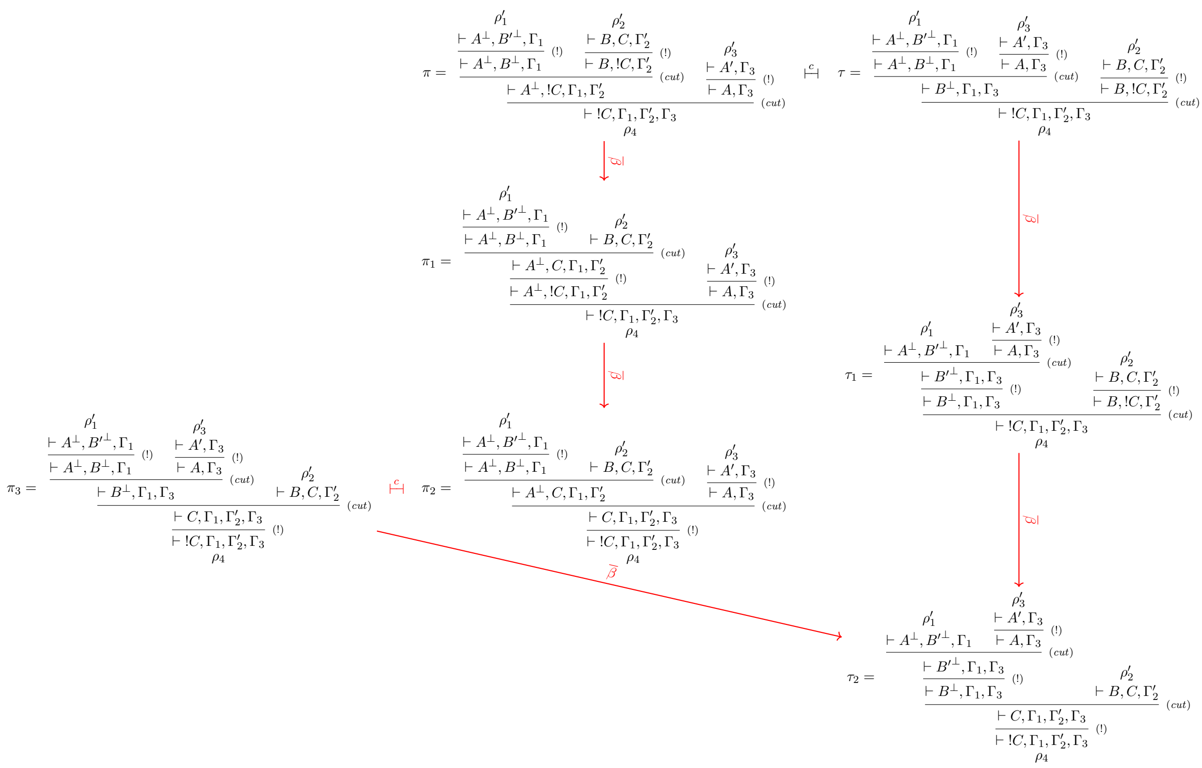


Figure 14: Representation of case *4* in the proof of Lemma 18

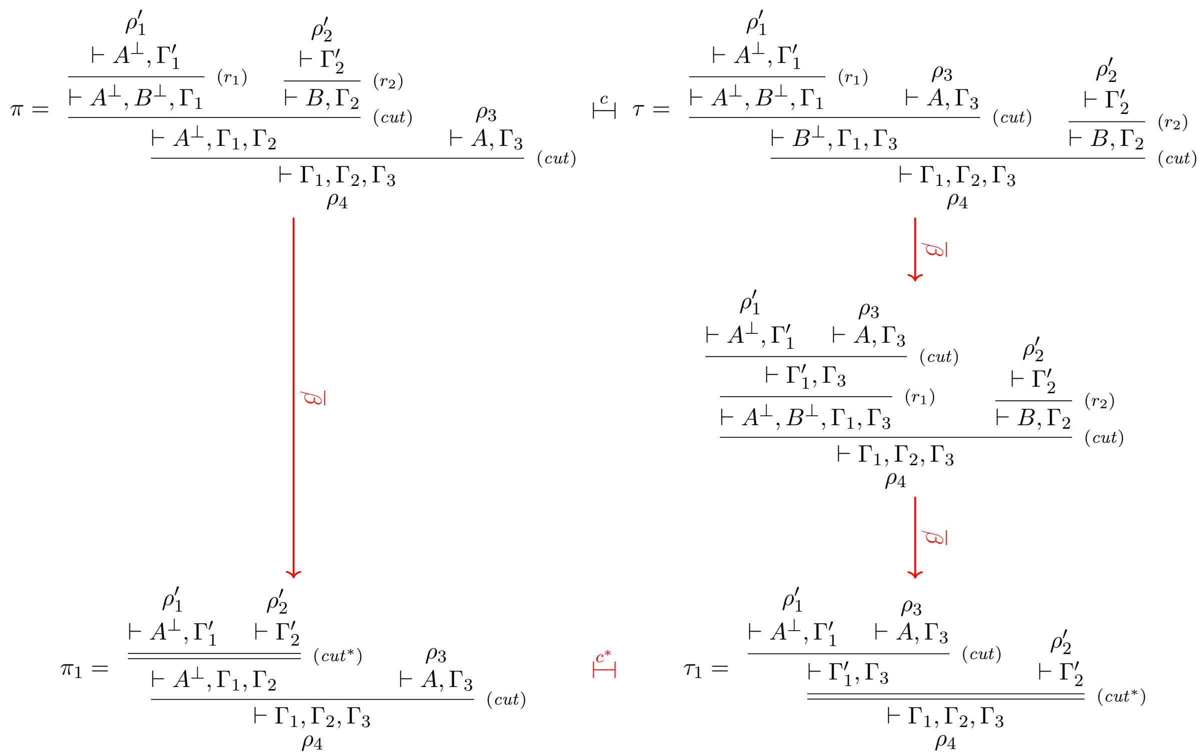


Figure 15: Representation of case 5 when neither r_1 nor r_2 is a $?_c$- or $?_w$-rule in the proof of Lemma 18

4.4 Church-Rosser modulo

We can now put everything together by applying Proposition 11 to obtain that $\xrightarrow{\overline{\beta}}$ is Church-Rosser modulo $(\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv})^*$, and deduce that cut-elimination $\xrightarrow{\beta}$ is Church-Rosser modulo rule commutation $\overset{r^*}{\vdash\!\dashv}$.

Proposition 20. *Cut-elimination* $\xrightarrow{\beta}$ *is Church-Rosser modulo rule commutation* $\overset{r^*}{\vdash\!\dashv}$.

Proof. One obtains that $\xrightarrow{\overline{\beta}}$ is Church-Rosser modulo $\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv}$ by applying Proposition 11 instantiated with $\to := \xrightarrow{\overline{\beta}}$, $\vdash\!\dashv := \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}$, and $\rightsquigarrow := \overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top}$, and whose hypotheses are given in Theorem 12 and Lemmas 14, 15 and 18. Let us consider two derivations π and ϕ such that $\pi\ (\xrightarrow{\beta} \cup \xleftarrow{\beta} \cup \overset{r}{\vdash\!\dashv})^*\ \phi$. By Corollary 13, there exist cut-free derivations π' and ϕ' such that $\pi \xrightarrow{\beta^*} \pi'$ and $\phi \xrightarrow{\beta^*} \phi'$. In particular, $\pi'\ (\xrightarrow{\overline{\beta}} \cup \xleftarrow{\overline{\beta}} \cup \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv})^*\ \phi'$. Since $\xrightarrow{\overline{\beta}}$ is Church-Rosser modulo $\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv}$, and as π' and ϕ' are $\xrightarrow{\overline{\beta}}$-normal forms, it follows that $\pi'\ (\overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv})^*\ \phi'$. But no $\overset{c}{\vdash\!\dashv}$ step can be applied on a cut-free derivation, and $\overset{r}{\vdash\!\dashv}$ preserves cut-freeness. Therefore, $\pi' \overset{r^*}{\vdash\!\dashv} \phi'$. Hence, cut-elimination $\xrightarrow{\beta}$ is Church-Rosser modulo rule commutation $\overset{r}{\vdash\!\dashv}$. $\square$

5 Equality up to cut-elimination and rule commutation

We prove here our main result: two derivations are equal up to cut-elimination $\xrightarrow{\beta}$ if and only if any of their $\xrightarrow{\beta}$-normal forms are related by $\overset{r^*}{\vdash\!\dashv}$. The direct way is obtained thanks to our Church-Rosser result, Proposition 20. The converse is a simple (but tedious) case study to obtain that rule commutation is included in equality up to cut-elimination.

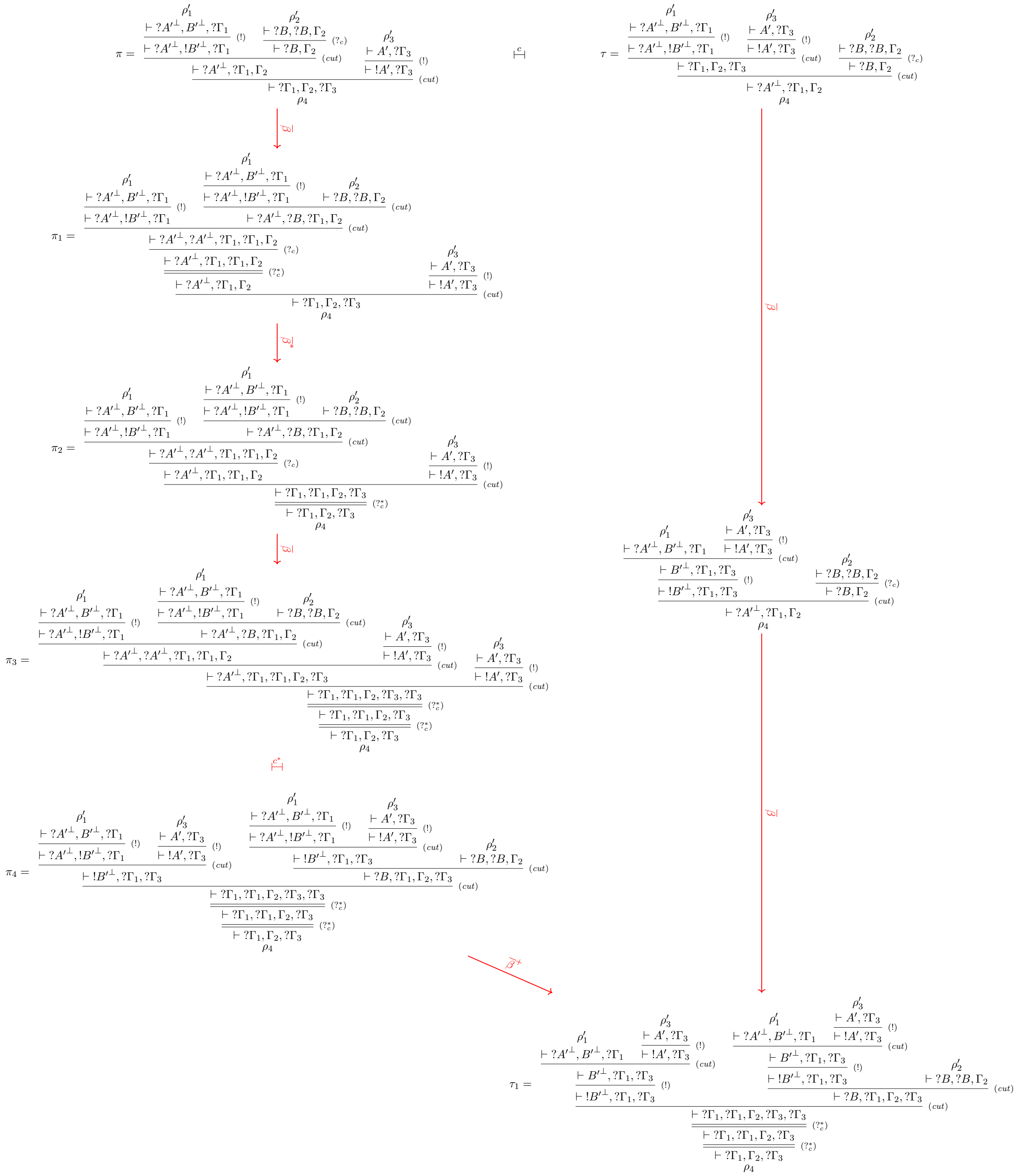


Figure 16: Representation of case 5 when r_2 is a $?_c$-rule in the proof of Lemma 18

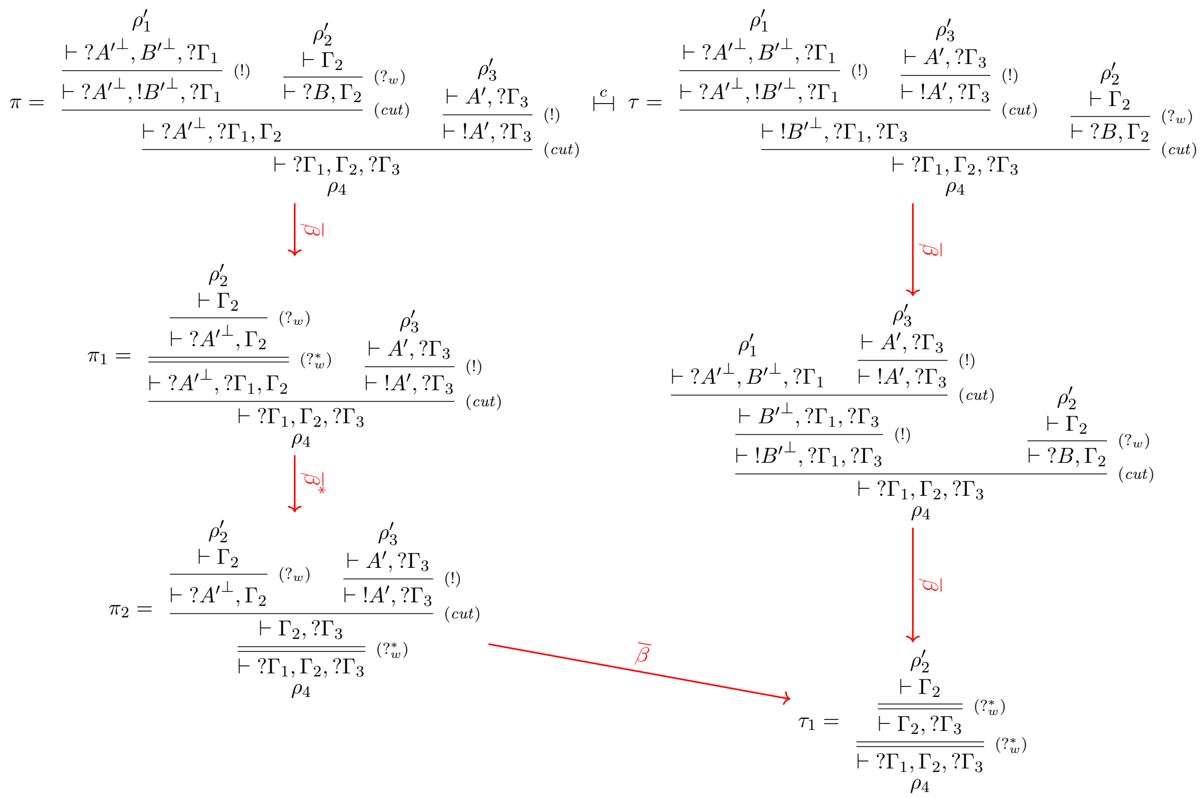


Figure 17: Representation of case 5 when r_2 is a $?_w$-rule in the proof of Lemma 18

We start by the latter, and demonstrate $\overset{r}{\longmapsto\!\!\dashv} \subseteq =_\beta$ (Section 5.1). Finally, we obtain that cut-elimination is the same as rule commutation on cut-free derivations (Section 5.2). We also deduce such a result for many sub-systems of linear logic (Section 5.3).

5.1 Rule commutation is included in equality up to cut-elimination

A first result, easy but tedious to prove, is that rule commutation is included in equality up to cut-elimination—and we do not even need to use the $cut - cut$ commutative case.

Proposition 21 ($\overset{r}{\longmapsto\!\!\dashv} \subseteq =_\beta$). *Given derivations π and τ, if $\pi \overset{r}{\longmapsto\!\!\dashv} \tau$ then $\pi \overset{\beta+}{\longleftarrow} \cdot \overset{\beta+}{\longrightarrow} \tau$.*

Proof. It suffices, for each rule commutation $\pi \overset{r}{\longmapsto\!\!\dashv} \tau$, to give a derivation ϕ with a single *cut*-rule and on which two commutative cut-elimination cases can be applied, such that applying a commutative step with its left premise (and then some more $\overset{\beta}{\longrightarrow}$ steps) yields π, whereas applying a commutative step with its right premise (and then some more $\overset{\beta}{\longrightarrow}$ steps) yields τ. Since all cases are similar, and there are 152 cases, we give here only a few representative ones—with links to the interactive linear logic prover C1ick ⅋ c⊗LLec⊥ where cut-elimination steps on the given ϕ can be followed. For completeness sake, an exhaustive proof with each of the 152 cases is given in Appendix D.

- $\& - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{}{\vdash \top, 0}(\top) \qquad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}(\top) \quad \dfrac{}{\vdash B, \top, \Gamma}(\top)}{\vdash A \,\&\, B, \top, \Gamma}(\&)}{\vdash A \,\&\, B, \top, \Gamma}(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{}{\vdash A \,\&\, B, \top, \Gamma}(\top) \qquad\qquad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}(\top) \quad \dfrac{}{\vdash B, \top, \Gamma}(\top)}{\vdash A \,\&\, B, \top, \Gamma}(\&)$$

- $\parr - \&$ *commutation* (See this cut-elimination in C1ick $\parr$ c$\otimes$LLec$\perp$ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}(\parr) \quad \dfrac{\overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}(\parr)}{\vdash A \parr B, C \,\&\, D, \Gamma}(\&) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}(ax) \quad \dfrac{}{\vdash A^{\perp}, A}(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}(\parr)}{\vdash A \parr B, C \,\&\, D, \Gamma}(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}(\parr) \quad \dfrac{\overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}(\parr)}{\vdash A \parr B, C \,\&\, D, \Gamma}(\&) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma} \quad \overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A, B, C \,\&\, D, \Gamma}(\&)}{\vdash A \parr B, C \,\&\, D, \Gamma}(\parr)$$

- $?_w - ?_w$ *commutation* (See this cut-elimination in C1ick $\parr$ c$\otimes$LLec$\perp$ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot)}{\vdash \bot, ?B, \Gamma}(?_w) \qquad \dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1, ?A}(?_w)}{\vdash ?A, ?B, \Gamma}(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}(?_w)}{\vdash ?A, ?B, \Gamma}(?_w) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?B, \Gamma}(?_w)}{\vdash ?A, ?B, \Gamma}(?_w)$$

- $?_c - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{}{\vdash B[C/X], (B[C/X])^{\perp}}(ax)}{\vdash \exists X B, (B[C/X])^{\perp}}(\exists) \qquad \dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, B[C/X], \Gamma}(?_c)}{\vdash ?A, \exists X B, \Gamma}(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, ?A, B[C/X], \Gamma}(?_c)}{\vdash ?A, \exists X B, \Gamma}(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, ?A, \exists X B, \Gamma}(\exists)}{\vdash ?A, \exists X B, \Gamma}(?_c)$$

□

5.2 Equality up to cut-elimination

We can now put together everything we proved to obtain our main contribution, *i.e.* the theorem claimed in the introduction: two derivations are equal up to cut-elimination if and only if any of their $\xrightarrow{\beta}$-normal forms are related by rule commutations. This will follows, as stated beforehand, from the following Church-Rosser modulo result.

Theorem 22 (Rule commutation is the core of cut-elimination)**.** *Two derivations are equal up to cut-elimination* $\xrightarrow{\beta}$ *if and only if any of their* $\xrightarrow{\beta}$*-normal forms are related by rule commutations* $\overset{r^*}{\longmapsto\!\dashv}$.

Proof. Consider two derivations π and ϕ, with π' and ϕ' two of their respective $\xrightarrow{\beta}$-normal forms. Observe $\pi =_\beta \phi \iff \pi' =_\beta \phi'$.

- If $\pi' =_\beta \phi'$ then, using Proposition 20, $\pi' \overset{r^*}{\longmapsto\!\dashv} \phi'$ since no $\xrightarrow{\beta}$ step can be applied on them.
- If $\pi' \overset{r^*}{\longmapsto\!\dashv} \phi'$, then $\pi' =_\beta \phi'$ thanks to Proposition 21. □

5.3 Equality up to cut-elimination in sub-systems of linear logic

Looking at our proofs, and as stated in the introduction, we also obtain that equality up to cut-elimination between cut-free proofs is the same as equality up to rule commutations in many sub-systems of linear logic. A **sub-system** of linear logic is a restriction keeping only some connectives and rules, such that the $⅋$-rule belongs to the sub-sytem if and only if the $\otimes$-rule does, and similarly for the $\perp$- and 1-rules, the $\&$- $\oplus_1$- and $\oplus_2$-rules, and the $\forall$- and $\exists$-rules. A sub-system is equipped with a cut-elimination procedure and with rule commutations: the ones for the full system restricted to its rules.

Proposition 23. *In any sub-system of linear logic, cut-elimination* $\xrightarrow{\beta}$ *is Church-Rosser modulo rule commutation* $\overset{r}{\longmapsto\!\dashv}$.

Proof. Same as Proposition 20, using that the proofs of Theorem 12 and Lemmas 14, 15 and 18 still hold when restricted to sub-systems. □

We *do not* have an equivalent of Proposition 21 for *all* sub-systems. Nonetheless, the exception are "unusal" sub-systems, such as additive-exponential linear logic with the $?_c$- and $?_w$-rules.

Proposition 24. *We have* $\overset{r}{\longmapsto\!\dashv} \subseteq =_\beta$ *in sub-systems of linear logic such that:*

- *if the* $?_c$*-rule belongs to the sub-system, then the* $⅋$*- and* $\otimes$*-rules also belong to it;*
- *if the* $?_w$*- or* mix_2*-rules belong to the sub-system, then the* $\perp$*- and* 1*-rules also belong to it;*
- *if the* $\cup$*-rule belongs to the sub-system, then the* $\perp$*-* 1*-* $\&$*-* $\oplus_1$*- and* $\oplus_2$*-rules also belong to it.*

Proof. See the proof of Proposition 21. □

We conclude a restriction of Theorem 22 in most sub-systems of linear logic, including *e.g.* propositional linear logic (with everything except quantifiers and optional rules), multiplicative linear logic (with or without units or quantifiers), multiplicative-exponential linear logic (with units and with or without quantifiers), and multiplicative-additive linear logic (with or without units or quantifiers).

Theorem 25. *Two derivations are equal up to cut-elimination $\xrightarrow{\beta}$ if and only if any of their $\xrightarrow{\beta}$-normal forms are related by rule commutations $\overset{r^*}{\vdash\!\dashv}$, in all sub-systems of linear logic such that:*

- *if the $?_c$-rule belongs to the sub-system, then the* ⅋- *and* $\otimes$*-rules also belong to it;*
- *if the $?_w$- or mix_2-rules belong to the sub-system, then the* $\bot$- *and* 1*-rules also belong to it;*
- *if the* $\cup$*-rule belongs to the sub-system, then the* $\bot$- 1- $\&$- $\oplus_1$- *and* $\oplus_2$*-rules also belong to it.*

Proof. Similar to Theorem 22, using Proposition 23 instead of Proposition 20 and Proposition 24 instead of Proposition 21. $\square$

6 Adding Rétoré transformations

We now add the Rétoré transformation $\xrightarrow{R}$ to our main results, Proposition 20 and Theorem 22. This yields that, between cut-free derivations, equality up to cut-elimination and Rétoré transformation is the same as equality up to rule commutation and Rétoré transformation. It suffices to close some diagrams involving the newly added rules and cut-elimination.

Lemma 26. *Let π, π_1 and π_2 be derivations such that $\pi_1 \xleftarrow{R} \pi \xrightarrow{\overline{\beta}} \pi_2$. Then $\pi_1 \xrightarrow{\overline{\beta}^=} \cdot \xleftarrow{R^*} \cdot \xleftarrow{\overline{\beta}^*} \cdot \overset{c^=}{\vdash\!\dashv} \pi_2$.*

Proof. If the two rewriting steps share no rule, then $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xleftarrow{R^*} \pi_2$ by applying these steps in the other order—taking care that if the $\xrightarrow{\overline{\beta}}$ step duplicates the rules of the $\xleftarrow{R}$ step then we have to apply it twice, and that we do not have to apply it if its rules are erased by the $\xrightarrow{\overline{\beta}}$ step. Otherwise, we distinguish cases according to the kind of $\pi \xrightarrow{\overline{\beta}} \pi_2$.

- If it is a $mix_2 - cut$ commutative step, then the $\xrightarrow{Rm}$ step can still be applied in π_2: $\pi_1 \xleftarrow{R} \pi_2$.
- If it is a $\cup - cut$ commutative step, then in π_2 we apply a $\emptyset - cut$ commutative case with the $\emptyset$-rule involved in the $\pi \xrightarrow{Ra} \pi_1$ step, after which we apply a $\xrightarrow{Ra}$ step. This yields $\pi_1 \xleftarrow{R} \cdot \xleftarrow{\overline{\beta}} \pi_2$.
- If it is a $?_c - cut$ commutative step, then in π_2 we apply a $?_w - cut$ commutative case with the $?_w$-rule involved in the $\pi \xrightarrow{Re} \pi_1$ step, after which we apply a $\xrightarrow{Re}$ step. This yields $\pi_1 \xleftarrow{R} \cdot \xleftarrow{\overline{\beta}} \pi_2$.
- If it is a $?_c - !$ key step, we obtain $\pi_1 \xleftarrow{R^*} \cdot \xleftarrow{\overline{\beta}^+} \cdot \overset{c^=}{\vdash\!\dashv} \pi_2$ by proceeding as illustrated on Figure 18. In π_2, we apply a $?_w - !$ key case (if needed first permuting the two introduced *cut*-rules), taking care that the $?_w$-rules introduced are put in the *reverse* ordering compared to the $?_c$ introduced in the $\pi \xrightarrow{\overline{\beta}} \pi_2$—*e.g.* if the latter introduce from top to bottom a $?_c$-rule on $?B_1$, etc., $?B_n$, we introduce from top to bottom $?_w$-rules on $?B_n$, etc., $?B_1$. These new $?_w$-rules commute with the *cut*-rule, using $?_w - cut$ commutative steps. We finally get π_1 by applying $\xrightarrow{Re}$ steps on the $?_w$- and $?_c$-rules introduced by our two key cases. $\square$

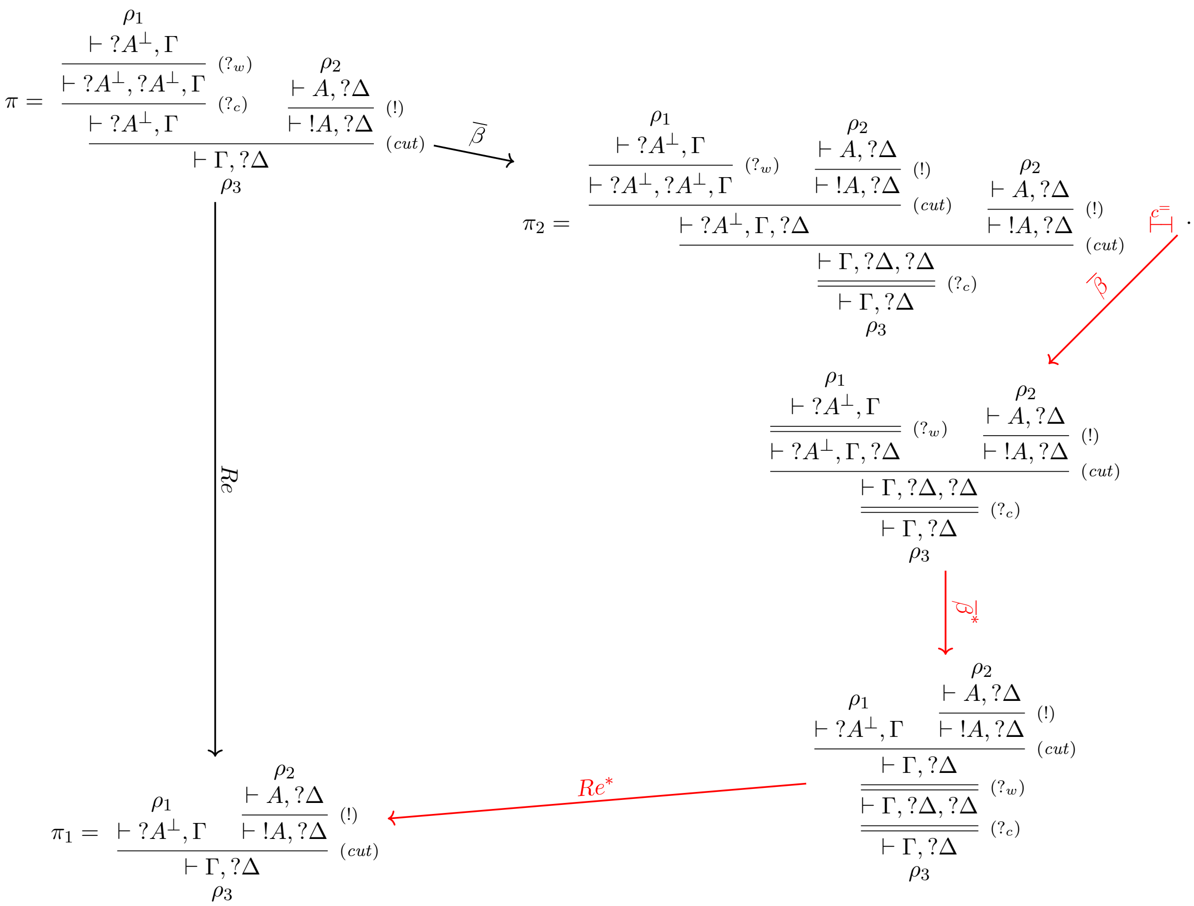


Figure 18: Representation of the last cases in the proofs of Lemma 26 and of Lemma 27

Lemma 27. *Let π, π_1 and π_2 be derivations such that $\pi_1 \xrightarrow{R} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$. Then $\pi_1 \xrightarrow{\overline{\beta}} \cdot \overset{c=}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}^*} \cdot \xrightarrow{R^*} \cdot \xleftarrow{\overline{\beta}} \pi_2$.*

Proof. Remark that the rules involved in $\pi_2 \xrightarrow{\overline{\beta}} \pi_3$ belong to π_1, for a $\xrightarrow{R}$ step creates no rule. If in π_1 we can apply the same $\xrightarrow{\overline{\beta}}$ step with these rules, then we obtain $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{R^*} \pi_3$ by doing so and then applying the $\xrightarrow{R}$ step—maybe twice if its rules have been duplicated by the $\pi_1 \xrightarrow{\overline{\beta}} \cdot$ step, or zero time if they have been erased. Otherwise, the $\xrightarrow{R}$ step is needed to do the cut-elimination step: the rules of this step are above a *cut*-rule. We distinguish cases.

If $\pi_1 \xrightarrow{Rm} \pi_2$. Here we have π_1 of the shape $\dfrac{\dfrac{\phi_1 \quad \dfrac{}{}\,(mix_0)}{}\,(mix_2) \quad \phi_2}{\phi_3}\,(cut)$ (or the symmetric cases with the mix_0-rule on the right branch of the mix_2-rule, or with the mix_2-rule on the right branch of the cut-rule). Observe that $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{Rm} \pi_2$ using a $mix_2 - cut$ commutative case:

$$\dfrac{\dfrac{\phi_1 \quad \dfrac{}{}\,(mix_0)}{}\,(mix_2) \quad \phi_2}{\phi_3}\,(cut) \xrightarrow{\overline{\beta}} \dfrac{\dfrac{\phi_1 \quad \phi_2}{}\,(cut) \quad \dfrac{}{}\,(mix_0)}{\phi_3}\,(mix_2) \xrightarrow{Rm} \dfrac{\phi_1 \quad \phi_2}{\phi_3}\,(cut)$$

If $\pi_1 \xrightarrow{Ra} \pi_2$. Here we have π_1 of the shape $\dfrac{\dfrac{\phi_1 \quad \dfrac{}{}\,(\emptyset)}{}\,(\cup) \quad \phi_2}{\phi_3}\,(cut)$ (or the symmetric cases with the $\emptyset$-rule on the right branch of the $\cup$-rule, or with the $\cup$-rule on the right branch of the cut-rule). Observe that $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \cdot \xrightarrow{Ra} \pi_2$ using a $\cup - cut$ commutative case followed by a $\emptyset - cut$ commutative case:

$$\dfrac{\dfrac{\phi_1 \quad \dfrac{}{}\,(\emptyset)}{}\,(\cup) \quad \phi_2}{\phi_3}\,(cut) \xrightarrow{\overline{\beta}} \dfrac{\dfrac{\phi_1 \quad \phi_2}{}\,(cut) \quad \dfrac{\dfrac{}{}\,(\emptyset) \quad \phi_2}{}\,(cut)}{\phi_3}\,(\cup) \xrightarrow{\overline{\beta}} \dfrac{\dfrac{\phi_1 \quad \phi_2}{}\,(cut) \quad \dfrac{}{}\,(\emptyset)}{\phi_3}\,(\cup) \xrightarrow{Ra} \dfrac{\phi_1 \quad \phi_2}{\phi_3}\,(cut)$$

If $\pi_1 \xrightarrow{Re} \pi_2$. Here we have π_1 of the shape $\dfrac{\dfrac{\dfrac{\phi_1}{}\,(?_w)}{}\,(?_c) \quad \phi_2}{\phi_3}\,(cut)$ (or the symmetric case with the $?_c$-rule on the right branch of the cut-rule). We have two sub-cases: either the cut-formula and the main formula of the $?_c$-rule are distinct, or they are the same occurrence.

In the first sub-case, we have $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{\overline{\beta}} \cdot \xrightarrow{Re} \pi_2$ using a $?_c - cut$ commutative case followed by a $?_w - cut$ commutative case:

$$\dfrac{\dfrac{\dfrac{\phi_1}{}\,(?_w)}{}\,(?_c) \quad \phi_2}{\phi_3}\,(cut) \xrightarrow{\overline{\beta}} \dfrac{\dfrac{\dfrac{\phi_1}{}\,(?_w) \quad \phi_2}{}\,(cut)}{\phi_3}\,(?_c) \xrightarrow{\overline{\beta}} \dfrac{\dfrac{\dfrac{\phi_1 \quad \phi_2}{}\,(cut)}{}\,(?_w)}{\phi_3}\,(?_c) \xrightarrow{Ra} \dfrac{\phi_1 \quad \phi_2}{\phi_3}\,(cut)$$

The second sub-case is the harder one. If the bottom rule of ϕ_2 commutes with the cut-rule in π_1, then it also commutes with it in π_2 and thus $\pi_1 \xrightarrow{\overline{\beta}} \cdot \xrightarrow{Re^*} \cdot \xleftarrow{\overline{\beta}} \pi_2$—with, as usual, the $\xrightarrow{Rm}$ step apply several times or not depending if its rules are duplicated or erased during the cut-elimination step. We can proceed similarly if the bottom rule of ϕ_2 is an ax-rule. Otherwise, the bottom rule of ϕ_2 is a $!$-rule, necessarily on the cut-formula. There, we proceed as on Figure 18, up to change of notations (π and π_1 on the figure are here respectively π_1 and π_2). Hence, $\pi_1 \xrightarrow{\overline{\beta}} \cdot \overset{c=}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}^+} \cdot \xrightarrow{Re^*} \pi_2$. ∎

Proposition 28. *Consider derivations π and ϕ, and π' and ϕ' two of their respective $\xrightarrow{\beta}$-normal forms. Then π $(\xrightarrow{\beta} \cup \xleftarrow{\beta} \cup \overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*$ ϕ if and only if π' $(\overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})$ ϕ'.*

Proof. We instantiate Proposition 11 with $\to := \xrightarrow{\overline{\beta}}$, $\vdash\!\dashv := \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv} \cup (\xrightarrow{R} \cup \xleftarrow{R})$, and $\rightsquigarrow := \overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \xrightarrow{\top} \cup \xrightarrow{R}$, whose hypotheses are proved in Theorem 12 and Lemmas 14, 15, 18, 26 and 27. Thus, $\xrightarrow{\overline{\beta}}$ is Church-Rosser modulo $\overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv} \cup (\xrightarrow{R} \cup \xleftarrow{R})$.

Let us consider two derivations π and ϕ, with respective $\xrightarrow{\beta}$-normal forms π' and ϕ'. Observe that $\pi\ (\xrightarrow{\beta} \cup \xleftarrow{\beta} \cup \overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi \iff \pi'\ (\xrightarrow{\beta} \cup \xleftarrow{\beta} \cup \overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi'$.

- Assume $\pi'\ (\xrightarrow{\beta} \cup \xleftarrow{\beta} \cup \overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi'$. By our Church-Rosser result, and using that π' and ϕ' are $\xrightarrow{\overline{\beta}}$-normal forms, we obtain $\pi'\ (\overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi'$. But no $\overset{c}{\vdash\!\dashv}$ step can be applied on a cut-free derivation, and $\overset{r}{\vdash\!\dashv}$, $\xrightarrow{R}$ and $\xleftarrow{R}$ all preserve cut-freeness. Therefore, $\pi'\ (\overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi'$.

- Assume $\pi'\ (\overset{r}{\vdash\!\dashv} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi'$. Thanks to Proposition 21, $\pi'\ (\xrightarrow{\overline{\beta}} \cup \xleftarrow{\overline{\beta}} \cup \xrightarrow{R} \cup \xleftarrow{R})^*\ \phi'$. $\square$

Please notice that it is easy to obtain from our proof equivalent of Proposition 28 in many sub-systems of linear logic, by proceeding exactly as in Section 5.3—and with the same constraints on sub-systems. Also, a result similar to Proposition 28 also holds when considering not all Rétoré transformations, but only some of them (*e.g.* taking only $\xrightarrow{Re}$). This is because in the proofs of Lemmas 26 and 27, the diagrams are closed using $\xrightarrow{Rm}$ (resp. $\xrightarrow{Ra}$, $\xrightarrow{Re}$) steps only if the hypotheses contained a $\xrightarrow{Rm}$ (resp. $\xrightarrow{Ra}$, $\xrightarrow{Re}$) step.

7 Adding axiom-expansion

Usually, one does not consider derivations as equal only up to cut-elimination, but also up to *axiom-expansion*. We can extend our main result in a framework with axiom-expansion; again, this is "mostly stable by sub-systems", and the presence or not of Rétoré transformations.

Theorem 29. *Let π_1 and π_2 be derivations, and set π_1' (resp. π_2') any $\xrightarrow{\beta}$-normal form of π_1 (resp. π_2), and π_1'' (resp. π_2'') any $\xrightarrow{\eta}$-normal form of π_1' (resp. π_2'). Then π_1 and π_2 are equal up to cut-elimination and axiom-expansion (*i.e. *$\pi_1 =_{\beta\eta} \pi_2$) if and only if $\pi_1'' \overset{r^*}{\vdash\!\dashv} \pi_2''$, with in this last sequence only $\xrightarrow{\eta}$-normal and $\xrightarrow{\beta}$-normal derivations.*

Remark 30. In the statement of Theorem 29, we need to first reach a cut-free derivation and then a derivation with all ax-rules expanded, for axiom-expansion preserves being cut-free but cut-elimination does not preserve having all ax-rules expanded—due to the key case for second order quantifiers.

The proof of Theorem 29 needs some intermediate results, and in particular that cut-elimination and axiom-expansion commute. The following technical result serves to prove this commutation claim.

Lemma 31. *Let* $\pi := \dfrac{\overset{\tau}{\vdash A, A^\perp} \qquad \overset{\phi_1}{\vdash A, \Gamma}}{\underset{\phi_2}{\vdash A, \Gamma}}\ (cut)$ *where* $\dfrac{}{\vdash A, A^\perp}\ (ax) \xrightarrow{\eta} \tau$. *Then, there exists a derivation ϕ_1' such that $\phi_1 \xrightarrow{\eta^*} \phi_1'$ and* $\pi\ (\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\beta})^+ \underset{\phi_2}{\overset{\phi_1'}{\vdash A, \Gamma}}$.

Proof. We proceed by induction on the size of the longest reduction sequence of π by $\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\beta}$ steps. This induction is well-founded since $\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\beta}$ is strongly normalizing, thanks to

Theorem 12, and the longest reduction exists since there is a finite number of derivations reachable by each $\overset{c^*}{\models\!\dashv} \cdot \xrightarrow{\overline{\beta}}$ step. We proceed by a case study on the last rule of ϕ_1.

1. If the last rule of ϕ_1 makes a commutative case with the cut-rule. In such a case, we apply this commutative cut-elimination step, yielding $\pi \xrightarrow{\overline{\beta}} \pi'$. The conclusion follows by applying the induction hypothesis on π'—remark we have to apply it twice if the step was a $\& - cut$ or $\cup - cut$ commutative step, and conclude directly if it was a $\top - cut$ or $\emptyset - cut$ commutative step. We assume from now on not to be in this case.

2. If the last rule of ϕ_1 is an ax-rule. We observe that $\pi \xrightarrow{\beta} \begin{array}{c}\tau\\ \vdash A, \Gamma\\ \phi_2\end{array}$ with $\phi_1 \xrightarrow{\eta} \tau$.

3. If the last rule of ϕ_1 is a $\parr$-rule. Here $A = B \parr C$ and

$$\pi = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash C, C^\perp}{\scriptstyle (ax)} \qquad \dfrac{}{\vdash B, B^\perp}{\scriptstyle (ax)}}{\vdash B, C, C^\perp \otimes B^\perp}{\scriptstyle (\otimes)}}{\vdash B \parr C, C^\perp \otimes B^\perp}{\scriptstyle (\parr)} \qquad \dfrac{\begin{array}{c}\phi_1^1\\ \vdash B, C, \Gamma\end{array}}{\vdash B \parr C, \Gamma}{\scriptstyle (\parr)}}{\begin{array}{c}\vdash B \parr C, \Gamma\\ \phi_2\end{array}}{\scriptstyle (cut)}$$

. By applying a $\parr - cut$ commutative case, then a $\parr - \otimes$ key case and two ax key cases, we find that $\pi \xrightarrow{\overline{\beta}^+} \dfrac{\begin{array}{c}\phi_1^1\\ \vdash B, C, \Gamma\end{array}}{\begin{array}{c}\vdash B \parr C, \Gamma\\ \phi_2\end{array}}{\scriptstyle (\parr)}$ as wanted.

4. If the last rule of ϕ_1 is a $\otimes$-rule. Here $A = B \otimes C$, $\Gamma = \Delta, \Sigma$ and

$$\pi = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}{\scriptstyle (ax)} \qquad \dfrac{}{\vdash C, C^\perp}{\scriptstyle (ax)}}{\vdash B \otimes C, C^\perp, B^\perp}{\scriptstyle (\otimes)}}{\vdash B \otimes C, C^\perp \parr B^\perp}{\scriptstyle (\parr)} \qquad \dfrac{\begin{array}{c}\phi_1^1\\ \vdash B, \Delta\end{array} \quad \begin{array}{c}\phi_1^2\\ \vdash C, \Sigma\end{array}}{\vdash B \otimes C, \Delta, \Sigma}{\scriptstyle (\otimes)}}{\begin{array}{c}\vdash B \otimes C, \Delta, \Sigma\\ \phi_2\end{array}}{\scriptstyle (cut)}$$

. Applying a $\parr - \otimes$ key case, followed by two $\otimes - cut$ commutative cases and two ax key cases, yields $\pi \xrightarrow{\overline{\beta}^+} \dfrac{\begin{array}{c}\phi_1^1\\ \vdash B, \Delta\end{array} \quad \begin{array}{c}\phi_1^2\\ \vdash C, \Sigma\end{array}}{\begin{array}{c}\vdash B \otimes C, \Delta, \Sigma\\ \phi_2\end{array}}{\scriptstyle (\otimes)}$.

5. If the last rule of ϕ_1 is a $\bot$- 1- &- $\oplus_1$- $\oplus_2$- $\top$- $?_d$- or !-rule. These cases are all similar to the previous two.

6. If the last rule of ϕ_1 is a $?_c$-rule. Here $A = ?B$ and

$$\pi = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}{\scriptstyle (ax)}}{\vdash ?B, B^\perp}{\scriptstyle (?_d)}}{\vdash ?B, !B^\perp}{\scriptstyle (!)} \qquad \dfrac{\begin{array}{c}\phi_1^1\\ \vdash ?B, ?B, \Gamma\end{array}}{\vdash ?B, \Gamma}{\scriptstyle (?_c)}}{\begin{array}{c}\vdash ?B, \Gamma\\ \phi_2\end{array}}{\scriptstyle (cut)}.$$

Applying a $?_c - !$ key case yields

$$\pi \xrightarrow{\overline{\beta}} \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}{\scriptstyle (ax)}}{\vdash ?B, B^\perp}{\scriptstyle (?_d)}}{\vdash ?B, !B^\perp}{\scriptstyle (!)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}{\scriptstyle (ax)}}{\vdash ?B, B^\perp}{\scriptstyle (?_d)}}{\vdash ?B, !B^\perp}{\scriptstyle (!)} \qquad \begin{array}{c}\phi_1^1\\ \vdash ?B, ?B, \Gamma\end{array}}{\vdash ?B, ?B, \Gamma}{\scriptstyle (cut)}}{\vdash ?B, ?B\Gamma}{\scriptstyle (cut)}}{\begin{array}{c}\vdash ?B, \Gamma\\ \phi_2\end{array}}{\scriptstyle (?_c)}$$

. We

conclude by two applications of the induction hypothesis.

7. If the last rule of ϕ_1 is a $?_w$-rule. Here $A = ?B$ and $\pi = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,{\scriptstyle (ax)}}{\vdash ?B, B^\perp}\,{\scriptstyle (?_d)}}{\vdash ?B, !B^\perp}\,{\scriptstyle (!)} \qquad \dfrac{\begin{matrix}\phi_1^1\\ \vdash \Gamma\end{matrix}}{\vdash ?B, \Gamma}\,{\scriptstyle (?_w)}}{\begin{matrix}\vdash ?B, \Gamma\\ \phi_2\end{matrix}}\,{\scriptstyle (cut)}$.

Applying a $?_w - !$ key case yields $\pi \xrightarrow{\beta} \dfrac{\begin{matrix}\phi_1^1\\ \vdash \Gamma\end{matrix}}{\begin{matrix}\vdash ?B, \Gamma\\ \phi_2\end{matrix}}\,{\scriptstyle (?_w)}$, hence we are done.

8. If the last rule of ϕ_1 is a $\forall$-rule. Here $A = \forall X B$ and

$\pi = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,{\scriptstyle (ax)}}{\vdash B, \exists X B^\perp}\,{\scriptstyle (\exists)}}{\vdash \forall X B, \exists X B^\perp}\,{\scriptstyle (\forall)} \qquad \dfrac{\begin{matrix}\phi_1^1\\ \vdash B, \Gamma\end{matrix}}{\vdash \forall X B, \Gamma}\,{\scriptstyle (\forall)}}{\begin{matrix}\vdash \forall X B, \Gamma\\ \phi_2\end{matrix}}\,{\scriptstyle (cut)}$. Applying a $\forall - cut$ commutative case, then a $\forall - \exists$ and an ax key cases yields $\pi \xrightarrow{\beta^+} \dfrac{\begin{matrix}\phi_1^1\\ \vdash B, \Gamma\end{matrix}}{\begin{matrix}\vdash \forall X B, \Gamma\\ \phi_2\end{matrix}}\,{\scriptstyle (\forall)}$ (up to α-renaming).

9. If the last rule of ϕ_1 is a $\exists$-rule. Here $A = \exists X B$ and

$\pi = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,{\scriptstyle (ax)}}{\vdash \exists X B, B^\perp}\,{\scriptstyle (\exists)}}{\vdash \exists X B, \forall X B^\perp}\,{\scriptstyle (\forall)} \qquad \dfrac{\begin{matrix}\phi_1^1\\ \vdash B[C/X], \Gamma\end{matrix}}{\vdash \exists X B, \Gamma}\,{\scriptstyle (\exists)}}{\begin{matrix}\vdash \exists X B, \Gamma\\ \phi_2\end{matrix}}\,{\scriptstyle (cut)}$. Applying a $\forall - \exists$ key case, then a $\exists - cut$ commutative case and an ax key case yields $\pi \xrightarrow{\beta^+} \dfrac{\begin{matrix}\phi_1^1\\ \vdash B[C/X], \Gamma\end{matrix}}{\begin{matrix}\vdash \exists X B, \Gamma\\ \phi_2\end{matrix}}\,{\scriptstyle (\exists)}$.

10. If the last rule of ϕ_1 is a cut-rule. In this last case, it is enough to commute up the *cut*-rule we are interested in as much as possible. Then, we are in one of the previous cases, allowing us to conclude. □

The abstract rewriting result we use to prove commutation is the following, discovered independently by Di Cosmo and Piperno [DP95, Lemma 3.2] and Geser [Ges90]. Remarkaly, it was used by Di Cosmo and Piperno in [DP95] to show commutation in second-order λ-calculus of η-expansion and β-reduction, so as to obtain confluence and strong normalization of the system with both.

Lemma 32 ([DP95, Lemma 3.2; Ges90])**.** *Let $\xrightarrow{x}$ and $\xrightarrow{y}$ be two relations. If $\xrightarrow{x}$ is strongly normalizing and $\xleftarrow{y} \cdot \xrightarrow{x} \subseteq \xrightarrow{x^+} \cdot \xleftarrow{y^*}$, then $\xleftarrow{y^*} \cdot \xrightarrow{x^*} \subseteq \xrightarrow{x^*} \cdot \xleftarrow{y^*}$ (i.e. $\xrightarrow{x}$ and $\xrightarrow{y}$ commute).*

Corollary 33. *The relations $\xrightarrow{\beta}$ and $\xrightarrow{\eta}$ commute: $\xleftarrow{\eta^*} \cdot \xrightarrow{\beta^*} \subseteq \xrightarrow{\beta^*} \cdot \xleftarrow{\eta^*}$.*

Proof. We will use this easily proved fact:

$$\xleftarrow{\eta} \cdot \overset{c}{\longmapsfrom\!\!\!\!\longmapsto} \subseteq \overset{c}{\longmapsfrom\!\!\!\!\longmapsto} \cdot \xleftarrow{\eta} \tag{1}$$

$$\begin{tikzcd}
\pi_2 \arrow[d, "\eta"] & \overset{c^*}{\vdash\!\dashv} & \pi_3 \arrow[rr, "\overline{\beta}"] \arrow[d, dashed, "\eta"] & & \pi_4 \arrow[d, dashed, "\eta^*"] & \overset{c^*}{\vdash\!\dashv} & \pi_5 \arrow[d, dashed, "\eta^*"] \\
\pi_1 & \overset{c^*}{\vdash\!\dashv} & \phi_1 & (\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}})^+ & \phi_2 & \overset{c^*}{\vdash\!\dashv} & \phi_3
\end{tikzcd}$$

Figure 19: Diagram for the proof of Corollary 33

We apply Lemma 32 to the relations $\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}} \cdot \overset{c^*}{\vdash\!\dashv}$ and $\xrightarrow{\eta}$, deducing that $\xleftarrow{\eta^*} \cdot (\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}} \cdot \overset{c^*}{\vdash\!\dashv})^* \subseteq (\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}} \cdot \overset{c^*}{\vdash\!\dashv})^* \cdot \xleftarrow{\eta^*}$. Please observe this, along with Equation (1), imply the wished $\xleftarrow{\eta^*} \cdot \xrightarrow{\beta^*} \subseteq \xrightarrow{\beta^*} \cdot \xleftarrow{\eta^*}$ (simply by doing a disjonction on whether $\xrightarrow{\beta^*}$ contains a $\xrightarrow{\overline{\beta}}$ step or not).

The relation $\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}} \cdot \overset{c^*}{\vdash\!\dashv}$ is strongly normalizing by Theorem 12. Consider derivations such that $\pi_1 \xleftarrow{\eta} \pi_2 \overset{c^*}{\vdash\!\dashv} \pi_3 \xrightarrow{\overline{\beta}} \pi_4 \overset{c^*}{\vdash\!\dashv} \pi_5$. Our reasoning is drawn on Figure 19. By repeated applications of Equation (1), there exists ϕ_1 such that $\pi_1 \overset{c^*}{\vdash\!\dashv} \phi_1 \xleftarrow{\eta} \pi_3$. If $\pi_3 \xrightarrow{\overline{\beta}} \pi_4$ is not an ax-key case on the ax-rule expanded in $\phi_1 \xleftarrow{\eta} \pi_3$, then we easily obtain a derivation ϕ_2 such that $\phi_1 \xrightarrow{\overline{\beta}} \phi_2 \xleftarrow{\eta^*} \pi_4$ (taking care the ax-rule may be duplicated or erased); otherwise, by Lemma 31, we obtain ϕ_2 such that ϕ_1 $(\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}})^+$ $\phi_2 \xleftarrow{\eta^*} \pi_4$. Then, we again apply repeatedly Equation (1) to get a derivation ϕ_3 such that $\phi_2 \overset{c^*}{\vdash\!\dashv} \phi_3 \xleftarrow{\eta^*} \pi_5$. Hence $\pi_1 \overset{c^*}{\vdash\!\dashv} \phi_1$ $(\overset{c^*}{\vdash\!\dashv} \cdot \xrightarrow{\overline{\beta}})^+$ $\phi_2 \overset{c^*}{\vdash\!\dashv} \phi_3 \xleftarrow{\eta^*} \pi_5$, concluding the proof. □

Proof of Theorem 29. Please remark that $\pi_1 =_{\beta\eta} \pi_2 \iff \pi_1'' =_{\beta\eta} \pi_2''$. The converse way is easy: if $\pi_1'' \overset{r^*}{\vdash\!\dashv} \pi_2''$, then $\pi_1'' =_{\beta\eta} \pi_2''$ by Proposition 21.

Assume $\pi_1'' =_{\beta\eta} \pi_2''$. We will use two well-known, and easily provable, facts about $\xrightarrow{\eta}$: it is strongly normalizing and confluent (even stronger: it has the diamond property); see *e.g.* [Di24, Section 2.2].

We first prove that π_1'' $(\overset{r}{\vdash\!\dashv} \cup \xrightarrow{\eta} \cup \xleftarrow{\eta})^*$ π_2''. Our reasoning is depicted on Figure 20. The sequence $\pi_1'' =_{\beta\eta} \pi_2''$ can be written as $\pi_1'' =_\beta \cdot =_\eta \cdot =_\beta \cdot =_\eta \cdot =_\beta \cdot =_\eta \dots \pi_2''$—in black on Figure 20. On the one hand, as $\xrightarrow{\eta}$ is strongly normalizing and confluent, $=_\eta \subseteq \xrightarrow{\eta^*} \cdot \xleftarrow{\eta^*}$ (simply by reducing both sides to their unique $\xrightarrow{\eta}$-normal form)—in blue on Figure 20. On the other hand, $=_\beta \subseteq \xrightarrow{\overline{\beta}^*} \cdot \overset{r^*}{\vdash\!\dashv} \cdot \xleftarrow{\overline{\beta}^*}$ by Theorem 22—in red on the graph, taking into account that π_1'' and π_2'' are cut-free derivations. We then apply Corollary 33 with $\xleftarrow{\eta^*} \cdot \xrightarrow{\beta^*} \subseteq \xrightarrow{\beta^*} \cdot \xleftarrow{\eta^*}$—in green on Figure 20. Finally, since our $\overset{r^*}{\vdash\!\dashv}$ steps were between cut-free derivations and $\xrightarrow{\eta}$ steps trivialy preserve being cut-free, it suffices to apply Theorem 22 again to find π_1'' $(\overset{r}{\vdash\!\dashv} \cup \xrightarrow{\eta} \cup \xleftarrow{\eta})^*$ π_2''—in violet on Figure 20.

Now that π_1'' $(\overset{r}{\vdash\!\dashv} \cup \xrightarrow{\eta} \cup \xleftarrow{\eta})^*$ π_2'', we prove $\pi_1'' \overset{r^*}{\vdash\!\dashv} \pi_2''$. One easily proves the following, simply because no $\overset{r}{\vdash\!\dashv}$ step involve an ax-rule (but may duplicate/superimpose or create/delete one):

$$\overset{r}{\vdash\!\dashv} \cdot \xrightarrow{\eta} \subseteq \xrightarrow{\eta^*} \cdot \overset{r}{\vdash\!\dashv} \cdot \xleftarrow{\eta^*} \tag{2}$$

By repeated applications of Equation (2) and strong normalization of $\xrightarrow{\eta}$, it is not complicated to deduce that if $\phi_1 \overset{r}{\vdash\!\dashv} \phi_2$ then $\phi_1 \xrightarrow{\eta^*} \phi_1' \overset{r}{\vdash\!\dashv} \phi_2' \xleftarrow{\eta^*} \phi_2$, with ϕ_1' (resp. ϕ_2') the $\xrightarrow{\eta}$-normal form of ϕ_1 (resp. ϕ_2).[1] Hence, we can assume in our sequence that every $\overset{r}{\vdash\!\dashv}$ step is between two $\xrightarrow{\eta}$-normal forms. We are now ready to conclude: our sequence is of the shape $\pi_1'' \overset{r^*}{\vdash\!\dashv} \cdot =_\eta \cdot \overset{r^*}{\vdash\!\dashv}$

[1] Proceed by induction on $n_1 + n_2$ with n_1 (resp. n_2) the size of the longuest η-expansion starting from ϕ_1 (resp.

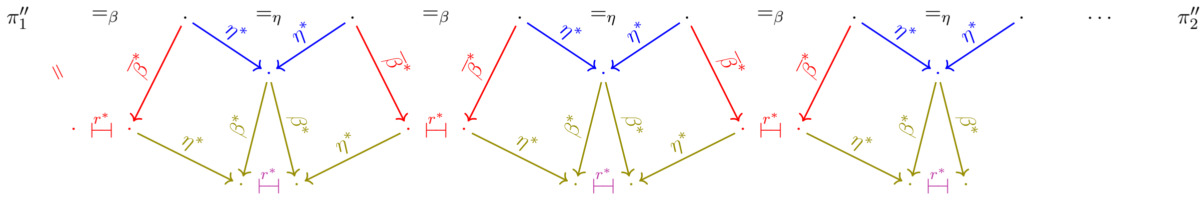


Figure 20: Diagram for the proof of Theorem 29

$\cdot =_\eta \ldots \pi_2''$ (remember π_1'' and π_2'' are $\xrightarrow{\eta}$-normal). But each $\overset{r}{\vdash\!\dashv}$ is between two $\xrightarrow{\eta}$-normal forms, and there is a unique $\xrightarrow{\eta}$-normal form per $=_\eta$-equivalence class. Hence, each $=_\eta$ step is the trivial equality by reflexivity, and $\pi_1'' \overset{r^*}{\vdash\!\dashv} \pi_2''$. ☐

8 Undecidability of equality up to rule commutation/cut-elimination

All this section consider as framework propositional linear logic, without quantifiers nor optional rules nor any of the additional transformation. We study three decision problems:

Equality up to Cut-elimination: Given two derivations, are they equal up to cut-elimination?

Proof Equivalence: Given two cut-free derivations, are they equal up to rule commutation?

Provability: Given a sequent, is it provable?

Thanks to our main result, Theorem 22, Equality up to Cut-elimination and Proof Equivalence are undecidable if and only if one of the two is (using Corollary 13 to eliminate all *cut*-rules by applying any maximal sequence of $\xrightarrow{\beta}$ steps). We prove they indeed are undecidable, by reducing Provability to Proof Equivalence (Lemma 35), using that Provability is well-known to be undecidable [Lin95]. Our reduction is simple and only uses the following elementary fact.

Fact 34. *A formula A is provable if and only if $!A$ is provable.*

Proof. If π is a derivation of A then we get a derivation of $!A$ using a $!$-rule: $\dfrac{\overset{\pi}{\vdash A}}{\vdash !A}\,(!)$. Conversely, if π is a derivation of $!A$ then we build a derivation of A: $\dfrac{\overset{\pi}{\vdash !A} \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash ?A^\perp, A}\,(?_d)}{\vdash A}\,(cut)$. ☐

The following encodes Provability into Proof Equivalence.

Lemma 35. *For any formula A:*

$$\left(\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r^*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_2)\right) \iff A \textit{ is provable}$$

ϕ_2). If it zero, then we are done. Otherwise, $\phi_1 \overset{r}{\vdash\!\dashv} \phi_2 \xrightarrow{\eta} \phi_2^1$ (up to switching ϕ_1 and ϕ_2). By Equation (2), $\phi_1 \xrightarrow{\eta^*} \phi_1^1 \overset{r}{\vdash\!\dashv} \phi_2^2 \xleftarrow{\eta^*} \phi_2^1 \xleftarrow{\eta} \phi_2$ for some ϕ_1^1 and ϕ_2^2. We conclude by induction hypothesis on $\phi_1^1 \overset{r}{\vdash\!\dashv} \phi_2^2$.

Proof. If A is provable, then both derivations are related to a same derivation by rule commutation:

$$\dfrac{\dfrac{}{\vdash !A \otimes \top_1, \top}\,(\top)}{\vdash !A \otimes \top_1, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\overset{\pi}{\vdash !A} \qquad \dfrac{}{\vdash \top_1, \top}\,(\top)}{\vdash !A \otimes \top_1, \top}\,(\otimes)}{\vdash !A \otimes \top_1, \top \oplus \top}\,(\oplus_i) \quad \text{(Fact 34 giving a derivation } \pi \text{ of } !A)$$

$$\overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\overset{\pi}{\vdash !A} \qquad \dfrac{}{\vdash \top_1, \top}\,(\top_1)}{\vdash !A \otimes \top_1, \top}\,(\otimes)}{\vdash !A \otimes \top_1, \top \oplus \top}\,(\oplus_i)$$

$$\overset{r}{\vdash\!\dashv} \quad \dfrac{\overset{\pi}{\vdash !A} \qquad \dfrac{\dfrac{}{\vdash \top_1, \top}\,(\top_1)}{\vdash \top_1, \top \oplus \top}\,(\oplus_i)}{\vdash !A \otimes \top_1, \top \oplus \top}\,(\otimes)$$

$$\overset{r}{\vdash\!\dashv} \quad \dfrac{\overset{\pi}{\vdash !A} \qquad \dfrac{}{\vdash \top_1, \top \oplus \top}\,(\top_1)}{\vdash !A \otimes \top_1, \top \oplus \top}\,(\otimes)$$

Thence:

$$\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r^*}{\vdash\!\dashv} \quad \dfrac{\overset{\pi}{\vdash !A} \qquad \dfrac{}{\vdash \top, \top \oplus \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\otimes) \quad \overset{r^*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_2)$$

If A is not provable, then we can compute the equivalence classes of both derivations:

$$\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash !A, \top}\,(\top) \qquad \dfrac{}{\vdash \top}\,(\top)}{\vdash !A \otimes \top, \top}\,(\otimes)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash !A, \top}\,(\top)}{\vdash !A, \top \oplus \top}\,(\oplus_i) \qquad \dfrac{}{\vdash \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\otimes)$$

This uses that there is no derivation of $!A$ by Fact 34. As these two classes are different, the two derivations are not related by rule commutations. ☐

Remark 36. Actually, if A is provable then all derivations of $!A \otimes \top, \top \oplus \top$ are related by $\overset{r^*}{\vdash\!\dashv}$, whereas if A is not provable there are two equivalence classes, with exactly those using the $\oplus_1$-rule and those using the $\oplus_2$-rule.

Remark 37. In Lemma 35, we use $!A$ instead of A so as to prevent commutations in $\dfrac{}{\vdash !A, \top}\,(\top)$, as the !-rule is the sole rule not commuting with the $\top$-rule. We can adapt our proof so as to not use the exponential ! up to some more technicalities. More precisely, we can prove that for any formula A and any atom X:

$$\left(\dfrac{\dfrac{}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top}\,(\top)}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r^*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top}\,(\top)}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top \oplus \top}\,(\oplus_2) \right) \iff A \text{ is provable}$$

Proposition 38 (Equality up to rule commutation is undecidable)**.** *Proof Equivalence in propositional linear logic is undecidable.*

Proof. It is known that Provability in propositional linear logic is undecidable [Lin95]. By Lemma 35, decidability of Proof Equivalence would entail decidability of the provability of any formula, hence of any sequent by taking the $\parr$ of its formulas. ☐

Sub-system	Complexity of Proof Equivalence
LL	Undecidable
unit-free MALL	LOGSPACE-complete [Bag17]
MLL	PSPACE-complete [HH16]
LL^2 without $\top$ and 0	decidable
MALL	decidable
MALL with $\top$ and 0 but not $\perp$ nor 1	PSPACE-hard
unit-free MLL	in P [HG16]
ALL	in P [Hei11]

Table 1: Complexity of Proof Equivalence in various sub-systems of linear logic

Proposition 39 (Equality up to cut-elimination is undecidable)**.** *Equality up to Cut-elimination in propositional linear logic is undecidable.*

Proof. Assume towards a contradiction that equality up to cut-elimination is decidable. Then, by Theorem 22, given two cut-free derivations one could decide whether they are equal up to rule commutation or not: contradiction with Proposition 38. ☐

Table 1 summarizes the complexity of Proof Equivalence for the sub-systems of linear logic where it is known. Let us detail it. The case of MLL (with units) was solved in [HH16], and unit-free MALL was solved in [Bag17]. We proved here undecidability in the case of propositional linear logic. Those are the systems exactly characterized. Some bounds for other systems are known. Using for MALL with additive units but no multiplicative units, and that provability of this system is PSPACE-complete—it is proved in that it is the case for MALL with all units, but units are irrelevant in its proof so that it also holds for MALL with additive units but no multiplicative units. Knowing that proof-nets identify derivations exactly up to rule commutation [HG16], the usual proof-nets for unit-free MLL allow to solve the problem in polynomial time. Proof-nets for ALL with units [Hei11] also solve it in P. Considering MALL, Proof Equivalence is decidable: there is a finite number of cut-free derivations, hence one can compute the equivalence class of a given derivation in finite time. A similar case is LL with second order quantifiers but no additive units: without commutations involving $\top$ (nor $\emptyset$), each equivalent class is finite because the number of rules in any of its derivations is bounded (but there may be infinitely many equivalence classes). Hence decidability of Proof Equivalence in this system.

Conclusion & Perspectives

This paper shows that cut-elimination in second-order linear logic is Church-Rosser modulo rule commutation, a result that was widely expected by the community but still not proved. Our proof also shows that this result holds in most sub-systems of linear logic, notably multiplicative linear logic (with and without units), additive linear logic (with and without units), multiplicative-additive linear logic (with and without units), multiplicative-exponential linear logic (with units), propositional linear logic (with units), and the second-order variants of all these systems. Furthermore, it can be extended to other usual transformation of derivations, such as axiom-expansion.

This key result implies that equality up to cut-elimination and equality up to rule commutation coincide on cut-free derivations, allowing to obtain complexity results on cut-elimination from the better studied and simpler equality up to rule commutation. In particular, we present a simple proof that equality up to rule commutation is undecidable in propositional linear logic. This is done through a reduction from provability, that is undecidable for propositional linear logic. Combined with our previous result, we deduce that whether two given derivations are equal up to cut-elimination is undecidable in propositional linear logic. This undecidability result completely prevents any canonical notion of proof-nets for propositional linear logic, as it implies either translating into a proof-net is undecidable, or equality of these proof-nets is undecidable—which in both cases means we would need a proof-net of infinite size to represent a (finite) derivation. Adding the *mix*-rule(s) does not change this undecidability result, for our proof directly extends to that setting (up to minor considerations due to the $\top - mix$ commutations). This contrasts with the case of multiplicative linear logic where adding the *mix*-rule turns the problem from PSPACE-complete [HH16] to solvable by proof-nets—see [FR94] for the first definition of these proof-nets.

A main tool to prove our main Church-Rosser modulo result is strong normalization of cut-elimination up to oriented rule commutations. We proved it building directly on a work by Michele Pagani and Lorenzo Tortora de Falco [PT10]. We conjecture a stronger result that the one we state holds, with less oriented rule commutations: see Appendix C for this conjecture.

A natural question is whether our main result linking cut-elimination and rule commutations has equivalents in variants of linear logic, *e.g.* differential linear logic [Ehr18] or linear logic with fixpoints [BDS16]. Such questions remain open, with as major difficulty that cut-elimination is particularly complicated in such systems—see *e.g.* [BS26].

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A Counter-examples to Strong Normalization up to Rule Commutation

We give here additional examples that $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\longmapsto}$ is not strongly normalizing. In each example given here, as well as the one from Section 3.1 using $\top - ?_c$ and $\top - ?_w$ commutations, one can replace $\otimes$-rules with mix_2-rules and/or $\top$-rules with $\emptyset$-rules to obtain other examples. Also, observe a counter-example similar to the one on Figure 5 arises by replacing the $\top - ?_c$ and $\top - ?_w$ commutations with a $\xrightarrow{Re}$ step, giving directly $\pi_1 \xrightarrow{Re} \pi_3$ and removing the need for $\top$.

With $\top - \otimes$ commutations Simply forbidding $\top - ?_c$ and $\top - ?_w$ commutations in the direction where they introduce rules is not enough. One can do similarly as in Section 3.1 by using a $\top - \otimes$ commutation introducing a new sub-derivation containing $?_c$-rules: see Figure 21(a) with a derivation π such that $\pi \xrightarrow{\overline{\beta}^+} \cdot \overset{r^+}{\longmapsto} \pi$.

With $\top - \exists$ commutations Second order quantifiers are also problematic with $\top$: Figure 21(b) presentes a derivation π such that $\pi \xrightarrow{\overline{\beta}^+} \cdot \overset{r^+}{\longmapsto} \pi$, the cut-elimination step being a $\forall - \exists$ key case and the rules commutations being $\top - \exists$ and $\top - \forall$ commutations.

With $\top - mix_2$ commutations and the mix_0- or $\emptyset$-rule On Figure 21(c) is depicted a counter-example with the mix_2- and mix_0-rules, where $\pi_1 \overset{r}{\longmapsto} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$ and π_1 is a sub-derivation of π_3, allowing to repeat these steps *ad nauseam*. The problem here is that a $\top - mix_2$ commutation can introduce some "noise" with the mix_0-rule: while we can apply an infinity of cut-elimination steps, the *cut*-rule is always on the same sequent. A similar example can be obtained by replacing the mix_0-rules with $\emptyset$-rules, still on the empty sequent.

With $\top - \cup$ commutations In presence of both $\top$- and $\cup$-rules, the relation $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\longmapsto}$ is also not strongly normalizing. A counter-example is given on Figure 21(d) where $\pi_1 \overset{r}{\longmapsto} \pi_2 \xrightarrow{\overline{\beta}} \pi_3$ and π_1 is a sub-derivation of π_3—it has even been duplicated!—so that these two steps can be repeated at will.

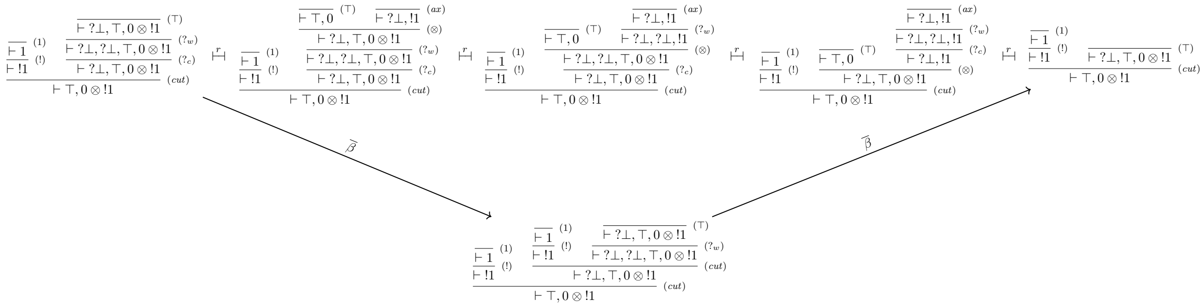
B Detailled proof of strong normalization

WORK IN PROGRESS...

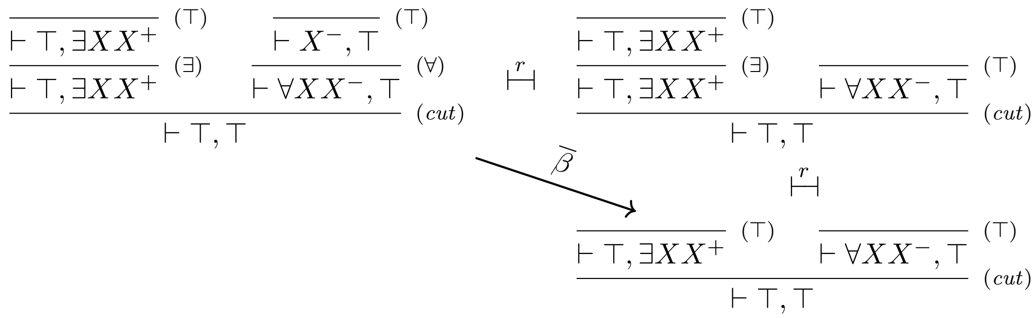
C Stronger conjecture for strong normalization up to oriented rule commutations

We present here a conjecture for a more general strong normalization result than the one we gave, Theorem 12. Remark the difference between Theorem 12 and the counter-examples from Section 3.1 and Appendix A is about what is to be considered a reduction and what a commutation, *i.e.* which commutations should be oriented and which should be taken in both directions. In particular, in Theorem 12 we remove all $C_\top^-$ and $C_\emptyset^-$ commutations. While Theorem 12 is enough for our purposes and allows to apply Proposition 11, we conjecture a stronger result holds with less commutations being oriented.

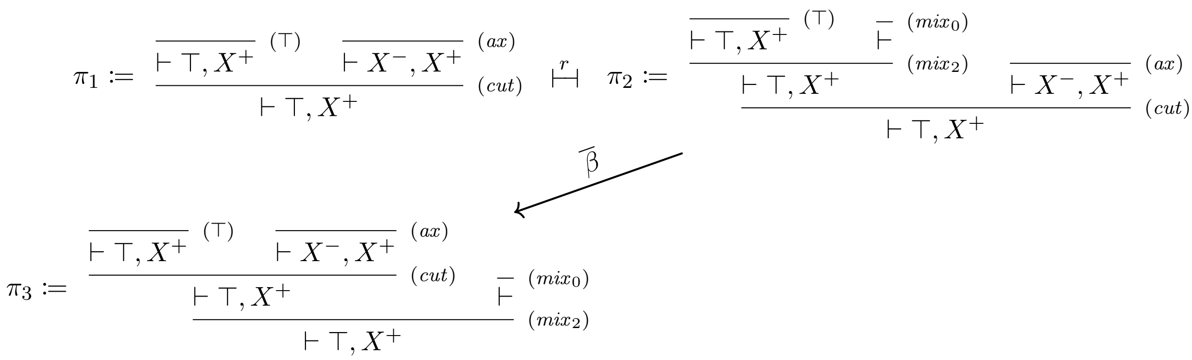
Conjecture 40 (Almost strong normalization of cut-elimination up to rule commutations)**.** *Set* $\xrightarrow{r\setminus\top}$ *rule commutations such that in a* $C_\top^\otimes$ *or* $C_\emptyset^\otimes$ *commutation the created derivation has no* $?_c$-*,*



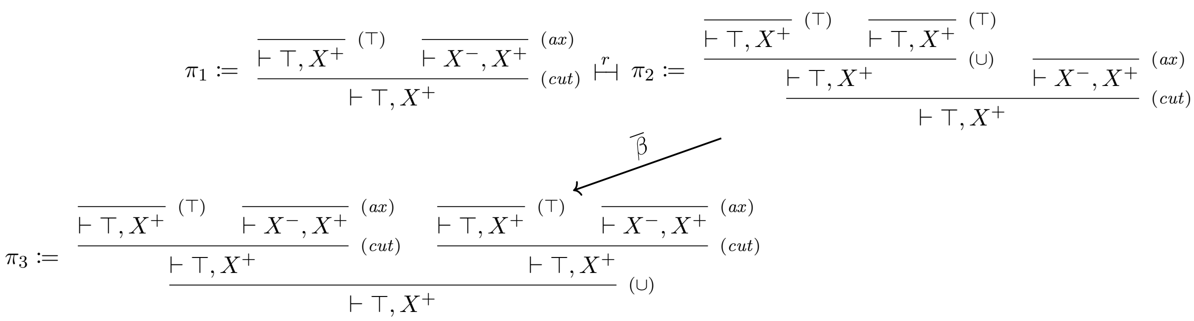
(a) With $\top - \otimes$ commutations



(b) With $\top - \exists$ commutations



(c) With $\top - mix_2$ commutations and the mix_0-rule



(d) With $\top - \cup$ commutations

Figure 21: Additional counter-examples to the strong normalization of $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$

mix_0- and $\cup$-rule, nor a $\emptyset$-rule applied on an empty sequent, and without the $C_{\top}^{?_c}$, $C_{\top}^{\exists}$, $C_{\top}^{\cup}$, $C_{\emptyset}^{?_c}$, $C_{\emptyset}^{\exists}$ and $C_{\emptyset}^{\cup}$ commutations. The relation $\xrightarrow{\overline{\beta}} \cdot (\overset{c}{\vdash\!\dashv} \cup \xrightarrow{r\backslash\top} \cup \xrightarrow{R})^*$ is strongly normalizing.

A stronger result than this conjecture should not hold looking at our counter-examples from Section 3.1 and Appendix A. Furthermore, this conjecture has been proved in the restricted case of propositional multiplicative-additive linear logic (with either the mix_2-rule or the mix_0-rule but not both of them) [Di24, Proposition 2.38]. Unfortunately, proving this conjecture seems particularly hard, and our tools inadequate: the paper [PT10] seems hardly adaptable to this case where some "creation" steps with the commutations of a $\top$-rule are allowed, because adding these non-erasing rules breaks needed confluence results, *e.g.* [PT10, Lemmas 4.12, 4.14, 4.16] (according to how we classify these new steps). In case the above conjecture is false, it is still possible to regard some more rule commutations as oriented to get other weaker conjectures that are still stronger than our main result on normalization, Theorem 12.

D Rule commutation is included in equality up to cut-elimination

We give here an exhaustive proof of Proposition 21, that was not written in the main part for the sake of clarity and brevity. It is given for completeness sake and is mostly computational content, better suited to be computer-checked in a proof assistant than read by a human being.

Proof of Proposition 21 from Page 35. We recall that we aim to give, for each rule commutation $\pi \overset{r}{\vdash\!\dashv} \tau$, a derivation ϕ with a single *cut*-rule and on which two commutative cut-elimination cases can be applied, such that applying a commutative step with its left premise (and then some more $\xrightarrow{\beta}$ steps) yields π, whereas applying a commutative step with its right premise (and then some more $\xrightarrow{\beta}$ steps) yields τ. Each of the 152 cases is given with a link to the interactive linear logic prover C1ick $\parr$ c$\otimes$LLec$\perp$ where cut-elimination steps on the given ϕ can be followed, except for cases involving quantifiers and optional rules (that are not in this prover at the time this article is written).

- $\parr - \parr$ *commutation* (See this cut-elimination in C1ick $\parr$ c$\otimes$LLec$\perp$ here.)

$$
\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, D, \Gamma}}{\vdash A \parr B, C, D, \Gamma}\,(\parr)}{\vdash A \parr B, C \parr D, \Gamma}\,(\parr)
\qquad
\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax) \quad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash B^\perp \otimes A^\perp, A, B}\,(\otimes)}{\vdash B^\perp \otimes A^\perp, A \parr B}\,(\parr)}{\vdash A \parr B, C \parr D, \Gamma}\,(cut)
$$

$$
\swarrow_{\beta^{\times}} \qquad\qquad \searrow_{\beta^{\times}}
$$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, D, \Gamma}}{\vdash A \parr B, C, D, \Gamma}\,(\parr)}{\vdash A \parr B, C \parr D, \Gamma}\,(\parr)
\qquad\qquad\qquad
\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, D, \Gamma}}{\vdash A, B, C \parr D, \Gamma}\,(\parr)}{\vdash A \parr B, C \parr D, \Gamma}\,(\parr)
$$

- $\parr - \otimes$ *commutations* (See these cut-eliminations in C1ick $\parr$ c$\otimes$LLec$\perp$ here and here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash C^{\perp}, C}\,(ax) \qquad \overset{\phi}{\vdash D, \Delta}}{\vdash C^{\perp}, C \otimes D, \Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash C, A \parr B, \Gamma}\,(\parr)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr) \qquad \overset{\phi}{\vdash D, \Delta}}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}\,(\otimes) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma} \qquad \overset{\phi}{\vdash D, \Delta}}{\vdash A, B, C \otimes D, \Gamma, \Delta}\,(\otimes)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}\,(\parr)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \dfrac{}{\vdash D^{\perp}, D}\,(ax)}{\vdash D^{\perp}, C \otimes D, \Gamma}\,(\otimes) \qquad \dfrac{\overset{\phi}{\vdash A, B, D, \Delta}}{\vdash D, A \parr B, \Delta}\,(\parr)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash A, B, D, \Delta}}{\vdash A \parr B, D, \Delta}\,(\parr)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}\,(\otimes) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \overset{\phi}{\vdash A, B, D, \Delta}}{\vdash A, B, C \otimes D, \Gamma, \Delta}\,(\otimes)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}\,(\parr)$$

- $\parr - \perp$ *commutation* (See this cut-elimination in C1ick $\parr$ c$\otimes$LLec$\perp$ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \perp}\,(\perp) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, B, \perp, \Gamma}\,(\perp)}{\vdash A \parr B, \perp, \Gamma}\,(\parr)}{\vdash A \parr B, \perp, \Gamma}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr)}{\vdash A \parr B, \perp, \Gamma}\,(\perp) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, B, \perp, \Gamma}\,(\perp)}{\vdash A \parr B, \perp, \Gamma}\,(\parr)$$

- $\parr - \&$ *commutation* (See this cut-elimination in C1ick $\parr$ c$\otimes$LLec$\perp$ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr) \qquad \dfrac{\overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}\,(\parr)}{\vdash A \parr B, C \mathbin{\&} D, \Gamma}\,(\&) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax) \qquad \dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}\,(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}\,(\parr)}{\vdash A \parr B, C \mathbin{\&} D, \Gamma}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr) \qquad \dfrac{\overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}\,(\parr)}{\vdash A \parr B, C \mathbin{\&} D, \Gamma}\,(\&) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma} \qquad \overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A, B, C \mathbin{\&} D, \Gamma}\,(\&)}{\vdash A \parr B, C \mathbin{\&} D, \Gamma}\,(\parr)$$

- $\parr - \oplus_1$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash C^{\perp}, C \oplus D}\,(\oplus_1) \qquad \dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, C \oplus D, \Gamma}\,(cut)$$

$$\swarrow \cancel{\beta} \qquad\qquad\qquad \cancel{\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, C \oplus D, \Gamma}\,(\oplus_1) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A, B, C \oplus D, \Gamma}\,(\oplus_1)}{\vdash A \parr B, C \oplus D, \Gamma}\,(\parr)$$

- $\parr - \oplus_2$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash D^{\perp}, D}\,(ax)}{\vdash D^{\perp}, C \oplus D}\,(\oplus_2) \qquad \dfrac{\overset{\pi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}\,(\parr)}{\vdash A \parr B, C \oplus D, \Gamma}\,(cut)$$

$$\swarrow \cancel{\beta} \qquad\qquad\qquad \cancel{\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}\,(\parr)}{\vdash A \parr B, C \oplus D, \Gamma}\,(\oplus_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, D, \Gamma}}{\vdash A, B, C \oplus D, \Gamma}\,(\oplus_2)}{\vdash A \parr B, C \oplus D, \Gamma}\,(\parr)$$

- $\parr - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{}{\vdash \top, 0}\,(\top) \qquad \dfrac{\dfrac{}{\vdash A, B, \top, \Gamma}\,(\top)}{\vdash A \parr B, \top, \Gamma}\,(\parr)}{\vdash A \parr B, \top, \Gamma}\,(cut)$$

$$\swarrow \cancel{\beta} \qquad\qquad\qquad \cancel{\beta} \searrow$$

$$\dfrac{}{\vdash A \parr B, \top, \Gamma}\,(\top) \qquad\qquad \dfrac{\dfrac{}{\vdash A, B, \top, \Gamma}\,(\top)}{\vdash A \parr B, \top, \Gamma}\,(\parr)$$

- $\parr - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash C^{\perp}, ?C}\,(?_d) \qquad \dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(cut)$$

$$\swarrow \cancel{\beta} \qquad\qquad\qquad \cancel{\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_d) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A, B, ?C, \Gamma}\,(?_d)}{\vdash A \parr B, ?C, \Gamma}\,(\parr)$$

- $\parr - ?_c$ *commutation* (See this cut-elimination in C1ick $\parr$ c⊗LLec⊥ here.)

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, ?C, ?C, \Gamma}}{\vdash A \parr B, ?C, ?C, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_c)
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax) \quad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A, B, B^\perp \otimes A^\perp}\,(\otimes)}{\vdash A \parr B, B^\perp \otimes A^\perp}\,(\parr)}{}
\;\Rightarrow\;
\dfrac{\vdash A \parr B, ?C, \Gamma \qquad \vdash A \parr B, B^\perp \otimes A^\perp}{\vdash A \parr B, ?C, \Gamma}\,(cut)
$$

$$
\swarrow_{\beta} \qquad\qquad \searrow_{\beta}
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, ?C, ?C, \Gamma}}{\vdash A \parr B, ?C, ?C, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_c)
\qquad\qquad
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, ?C, ?C, \Gamma}}{\vdash A, B, ?C, ?C, \Gamma}\,(?_c)}{\vdash A \parr B, ?C, \Gamma}\,(\parr)
$$

- $\parr - ?_w$ *commutation* (See this cut-elimination in C1ick $\parr$ c⊗LLec⊥ here.)

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_w)
\qquad
\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax) \quad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A, B, B^\perp \otimes A^\perp}\,(\otimes)}{\vdash A \parr B, B^\perp \otimes A^\perp}\,(\parr)
}{\vdash A \parr B, ?C, \Gamma}\,(cut)
$$

$$
\swarrow_{\beta} \qquad\qquad \searrow_{\beta}
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_w)
\qquad\qquad
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, \Gamma}}{\vdash A, B, ?C, \Gamma}\,(?_w)}{\vdash A \parr B, ?C, \Gamma}\,(\parr)
$$

- $\parr - \forall$ *commutation*

$$
\dfrac{
X \text{ not free in } A \parr B, \Gamma \;
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, \forall X C, \Gamma}\,(\forall)
\qquad
\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax) \quad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A, B, B^\perp \otimes A^\perp}\,(\otimes)}{\vdash A \parr B, B^\perp \otimes A^\perp}\,(\parr)
}{\vdash A \parr B, \forall X C, \Gamma}\,(cut)
$$

$$
\swarrow_{\beta} \qquad\qquad \searrow_{\beta}
$$

$$
X \text{ not free in } A \parr B, \Gamma \;
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, \forall X C, \Gamma}\,(\forall)
\qquad\qquad
X \text{ not free in } A, B, \Gamma \;
\dfrac{\dfrac{\dfrac{\pi}{\vdash A, B, C, \Gamma}}{\vdash A, B, \forall X C, \Gamma}\,(\forall)}{\vdash A \parr B, \forall X C, \Gamma}\,(\parr)
$$

- $\parr - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C[D/X], \Gamma}}{\vdash A \parr B, C[D/X], \Gamma}\,(\parr)}{\vdash A \parr B, \exists X C, \Gamma}\,(\exists) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax) \quad \dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}\,(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}\,(\parr)}{\vdash A \parr B, \exists X C, \Gamma}\,(cut)$$

$$\swarrow \beta \qquad\qquad \beta \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C[D/X], \Gamma}}{\vdash A \parr B, C[D/X], \Gamma}\,(\parr)}{\vdash A \parr B, \exists X C, \Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C[D/X], \Gamma}}{\vdash A, B, \exists X C, \Gamma}\,(\exists)}{\vdash A \parr B, \exists X C, \Gamma}\,(\parr)$$

- $\parr - mix_2$ *commutations*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr) \quad \overset{\phi}{\vdash \Delta}}{\vdash A \parr B, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax) \quad \dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}\,(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}\,(\parr)}{\vdash A \parr B, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta \qquad\qquad \beta \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr) \quad \overset{\phi}{\vdash \Delta}}{\vdash A \parr B, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A, B, \Gamma, \Delta}\,(mix_2)}{\vdash A \parr B, \Gamma, \Delta}\,(\parr)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, B, \Delta}}{\vdash A \parr B, \Delta}\,(\parr)}{\vdash A \parr B, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax) \quad \dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}\,(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}\,(\parr)}{\vdash A \parr B, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta \qquad\qquad \beta \searrow$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, B, \Delta}}{\vdash A \parr B, \Delta}\,(\parr)}{\vdash A \parr B, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, B, \Delta}}{\vdash A, B, \Gamma, \Delta}\,(mix_2)}{\vdash A \parr B, \Gamma, \Delta}\,(\parr)$$

- $\parr - \cup$ *commutation*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}(\parr) \quad \dfrac{\overset{\phi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}(\parr)}{\vdash A \parr B, \Gamma}(\cup) \quad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}(ax) \quad \dfrac{}{\vdash A^{\perp}, A}(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}(\parr)}{\vdash A \parr B, \Gamma, \Delta}(cut)$$

$\beta^{\times}$ (left) $\quad$ $\beta^{\times}$ (right)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}(\parr) \quad \dfrac{\overset{\phi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}(\parr)}{\vdash A \parr B, \Gamma}(\cup) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma} \quad \overset{\phi}{\vdash A, B, \Gamma}}{\vdash A, B, \Gamma}(\cup)}{\vdash A \parr B, \Gamma}(\parr)$$

- $\parr - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash A \parr B, \Gamma}(\emptyset) \quad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}(ax) \quad \dfrac{}{\vdash A^{\perp}, A}(ax)}{\vdash A, B, B^{\perp} \otimes A^{\perp}}(\otimes)}{\vdash A \parr B, B^{\perp} \otimes A^{\perp}}(\parr)}{\vdash A \parr B, \Gamma, \Delta}(cut)$$

$\beta^{\times}$ (left) $\quad$ $\beta^{\times}$ (right)

$$\dfrac{}{\vdash A \parr B, \Gamma}(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash A, B, \Gamma}(\emptyset)}{\vdash A \parr B, \Gamma}(\parr)$$

- $\otimes - \otimes$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here, here and here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \overset{\phi}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}(\otimes) \quad \dfrac{\dfrac{}{\vdash A^{\perp}, A}(ax) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, A \otimes B, \Sigma}(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(cut)$$

$\beta^{\times}$ (left) $\quad$ $\beta^{\times}$ (right)

$$\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, D, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \overset{\phi}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}(\otimes) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\tau}{\vdash D, \Sigma}}{\vdash A, C \otimes D, \Gamma, \Sigma}(\otimes) \quad \dfrac{\dfrac{}{\vdash A^{\perp}, A}(ax) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(cut)$$

$\beta^{\times}$ (left) $\quad$ $\beta^{\times}$ (right)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}(\otimes) \quad \overset{\tau}{\vdash D, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\tau}{\vdash D, \Sigma}}{\vdash A, C \otimes D, \Gamma, \Sigma}(\otimes) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash C,\Gamma} \qquad \dfrac{}{\vdash D^{\perp}, D}\,(ax)}{\vdash D^{\perp}, C\otimes D,\Gamma}\,(\otimes) \qquad \dfrac{\overset{\phi}{\vdash A,\Delta} \qquad \overset{\tau}{\vdash B,D,\Sigma}}{\vdash A\otimes B, D,\Delta,\Sigma}\,(\otimes)}{\vdash A\otimes B, C\otimes D,\Gamma,\Delta,\Sigma}\,(cut)$$

$$\swarrow_{\beta} \qquad\qquad \searrow_{\beta}$$

$$\dfrac{\overset{\pi}{\vdash C,\Gamma} \qquad \dfrac{\overset{\phi}{\vdash A,\Delta} \qquad \overset{\tau}{\vdash B,D,\Sigma}}{\vdash A\otimes B, D,\Gamma,\Sigma}\,(\otimes)}{\vdash A\otimes B, C\otimes D,\Gamma,\Delta,\Sigma}\,(\otimes) \qquad\qquad \dfrac{\overset{\phi}{\vdash A,\Delta} \qquad \dfrac{\overset{\pi}{\vdash C,\Gamma} \qquad \overset{\tau}{\vdash B,D,\Sigma}}{\vdash B, C\otimes D,\Delta,\Sigma}\,(\otimes)}{\vdash A\otimes B, C\otimes D,\Gamma,\Delta,\Sigma}\,(\otimes)$$

- $\otimes - \perp$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,\perp,\Gamma}\,(\perp) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A^{\perp}, A\otimes B,\Delta}\,(\otimes)}{\vdash A\otimes B,\perp,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta} \qquad\qquad \searrow_{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B,\perp,\Gamma,\Delta}\,(\perp) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,\perp,\Gamma}\,(\perp) \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,\perp,\Gamma,\Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B,\Delta}}{\vdash B,\perp,\Delta}\,(\perp) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A\otimes B,\Gamma}\,(\otimes)}{\vdash A\otimes B,\perp,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta} \qquad\qquad \searrow_{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B,\perp,\Gamma,\Delta}\,(\perp) \qquad\qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{\overset{\phi}{\vdash B,\Delta}}{\vdash B,\perp,\Delta}\,(\perp)}{\vdash A\otimes B,\perp,\Gamma,\Delta}\,(\otimes)$$

- $\otimes - \&$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$\dfrac{\dfrac{\overset{\phi}{\vdash B,C,\Delta} \qquad \overset{\tau}{\vdash B,D,\Delta}}{\vdash B, C\,\&\,D,\Delta}\,(\&) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A\otimes B,\Gamma}\,(\otimes)}{\vdash A\otimes B, C\,\&\,D,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta} \qquad\qquad \searrow_{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \overset{\phi}{\vdash B,C,\Delta}}{\vdash A\otimes B,C,\Gamma,\Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \overset{\tau}{\vdash B,D,\Delta}}{\vdash A\otimes B,D,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B, C\,\&\,D,\Gamma,\Delta}\,(\&) \qquad\qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{\overset{\phi}{\vdash B,C,\Delta} \qquad \overset{\tau}{\vdash B,D,\Delta}}{\vdash B, C\,\&\,D,\Delta}\,(\&)}{\vdash A\otimes B, C\,\&\,D,\Gamma,\Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\tau}{\vdash A, D, \Gamma}}{\vdash A, C \,\&\, D, \Gamma}\,(\&) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\tau}{\vdash A, D, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\tau}{\vdash A, D, \Gamma}}{\vdash A, C \,\&\, D, \Gamma}\,(\&) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes)$$

- $\otimes - \oplus_1$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_1) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_1) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_1) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, C \oplus D, \Delta}\,(\oplus_1) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_1) \qquad\qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, C \oplus D, \Delta}\,(\oplus_1)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$$

- $\otimes - \oplus_2$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_2) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$$

$$
\dfrac{\dfrac{\overset{\phi}{\vdash B, D, \Delta}}{\vdash B, C \oplus D, \Delta}\,(\oplus_2) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_2)
\qquad\qquad
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash B, D, \Delta}}{\vdash B, C \oplus D, \Delta}\,(\oplus_2)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)
$$

- $\otimes - \top$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$
\dfrac{\dfrac{}{\vdash \top, 0}\,(\top) \qquad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash B, \Delta}}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\top)
\qquad\qquad
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash B, \Delta}}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\otimes)
$$

$$
\dfrac{\dfrac{}{\vdash \top, 0}\,(\top) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B, \top, \Delta}\,(\top)}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\top)
\qquad\qquad
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B, \top, \Delta}\,(\top)}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\otimes)
$$

- $\otimes - ?_d$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_d) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A^\perp, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_d)
\qquad\qquad
\xrightarrow{\overline{\beta}^{\times}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_d) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)
$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, ?C, \Delta}\,(?_d) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(cut)$$

$$\overset{\beta^{\times}}{\swarrow} \qquad\qquad \overset{\beta^{\times}}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_d) \qquad\qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, ?C, \Delta}\,(?_d)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$$

- $\otimes - ?_c$ *commutation* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_c) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(cut)$$

$$\overset{\beta^{\times}}{\swarrow} \qquad\qquad \overset{\beta^{\times}}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, ?C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_c) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_c) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B, ?C, ?C, \Delta}}{\vdash B, ?C, \Delta}\,(?_c) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(cut)$$

$$\overset{\beta^{\times}}{\swarrow} \qquad\qquad \overset{\beta^{\times}}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, ?C, ?C, \Delta}}{\vdash A \otimes B, ?C, ?C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_c) \qquad\qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, ?C, ?C, \Delta}}{\vdash B, ?C, \Delta}\,(?_c)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$$

- $\otimes - ?_w$ *commutations* (See these cut-eliminations in C1ick ⅋ c⊗LLec⊥ here and here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_w) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(cut)$$

$$\overset{\beta^{\times}}{\swarrow} \qquad\qquad \overset{\beta^{\times}}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_w) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_w) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B,\Delta}}{\vdash B,?C,\Delta}\,(?_w) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{}{\vdash B^{\perp},B}\,(ax)}{\vdash B^{\perp},A\otimes B,\Gamma}\,(\otimes)}{\vdash A\otimes B,?C,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta^{\times}} \qquad\qquad \searrow^{\beta^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B,?C,\Gamma,\Delta}\,(?_w) \qquad\qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{\overset{\phi}{\vdash B,\Delta}}{\vdash B,?C,\Delta}\,(?_w)}{\vdash A\otimes B,?C,\Gamma,\Delta}\,(\otimes)$$

- $\otimes - \forall$ *commutations*

$$X \text{ not free in } A,\Gamma \quad \dfrac{\dfrac{\overset{\pi}{\vdash A,C,\Gamma}}{\vdash A,\forall XC,\Gamma}\,(\forall) \qquad \dfrac{\dfrac{}{\vdash A^{\perp},A}\,(ax) \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A^{\perp},A\otimes B,\Delta}\,(\otimes)}{\vdash A\otimes B,\forall XC,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta^{\times}} \qquad\qquad \searrow^{\beta^{\times}}$$

$$X \text{ not free in } A{\otimes}B,\Gamma,\Delta \quad \dfrac{\dfrac{\overset{\pi}{\vdash A,C,\Gamma} \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,C,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B,\forall XC,\Gamma,\Delta}\,(\forall) \qquad X \text{ not free in } A,\Gamma \quad \dfrac{\dfrac{\overset{\pi}{\vdash A,C,\Gamma}}{\vdash A,\forall XC,\Gamma}\,(\forall) \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,\forall XC,\Gamma,\Delta}\,(\otimes)$$

$$X \text{ not free in } B,\Delta \quad \dfrac{\dfrac{\overset{\phi}{\vdash B,C,\Delta}}{\vdash B,\forall XC,\Delta}\,(\forall) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{}{\vdash B^{\perp},B}\,(ax)}{\vdash B^{\perp},A\otimes B,\Gamma}\,(\otimes)}{\vdash A\otimes B,\forall XC,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta^{\times}} \qquad\qquad \searrow^{\beta^{\times}}$$

$$X \text{ not free in } A{\otimes}B,\Gamma,\Delta \quad \dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \overset{\phi}{\vdash B,C,\Delta}}{\vdash A\otimes B,C,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B,\forall XC,\Gamma,\Delta}\,(\forall) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad X \text{ not free in } B,\Delta \quad \dfrac{\overset{\phi}{\vdash B,C,\Delta}}{\vdash B,\forall XC,\Delta}\,(\forall)}{\vdash A\otimes B,\forall XC,\Gamma,\Delta}\,(\otimes)$$

- $\otimes - \exists$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,C[D/X],\Gamma}}{\vdash A,\exists XC,\Gamma}\,(\exists) \qquad \dfrac{\dfrac{}{\vdash A^{\perp},A}\,(ax) \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A^{\perp},A\otimes B,\Delta}\,(\otimes)}{\vdash A\otimes B,\exists XC,\Gamma,\Delta}\,(cut)$$

$$\swarrow_{\beta^{\times}} \qquad\qquad \searrow^{\beta^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,C[D/X],\Gamma} \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,C[D/X],\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B,\exists XC,\Gamma,\Delta}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A,C[D/X],\Gamma}}{\vdash A,\exists XC,\Gamma}\,(\exists) \qquad \overset{\phi}{\vdash B,\Delta}}{\vdash A\otimes B,\exists XC,\Gamma,\Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B, C[D/X], \Delta}}{\vdash B, \exists XC, \Delta}\,(\exists) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow_{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, C[D/X], \Delta}}{\vdash A \otimes B, C[D/X], \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}\,(\exists) \qquad\qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, C[D/X], \Delta}}{\vdash B, \exists XC, \Delta}\,(\exists)}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}\,(\otimes)$$

- $\otimes - mix_2$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash \Sigma}}{\vdash A, \Gamma, \Sigma}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow_{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash \Sigma}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash \Sigma}}{\vdash A, \Gamma, \Sigma}\,(mix_2) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, A \otimes B, \Sigma}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow_{\overline{\beta}^{\times}}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B, \Delta} \quad \overset{\tau}{\vdash \Sigma}}{\vdash B, \Delta, \Sigma}\,(mix_2) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(cut)$$

$$\swarrow_{\overline{\beta}^{\times}} \qquad\qquad \searrow_{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash \Sigma}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, \Delta} \quad \overset{\tau}{\vdash \Sigma}}{\vdash B, \Delta, \Sigma}\,(mix_2)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash B, \Gamma, \Sigma}\,(mix_2) \qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \qquad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(cut)$$

$$\overset{\not\beta^{+}}{\swarrow} \qquad\qquad\qquad \overset{\not\beta^{+}}{\searrow}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \qquad \dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash B, \Gamma, \Sigma}\,(mix_2)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}\,(\otimes)$$

- $\otimes - \cup$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}\,(cut)$$

$$\overset{\not\beta^{+}}{\swarrow} \qquad\qquad\qquad \overset{\not\beta^{+}}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\tau}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}\,(\cup) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)$$

$$\dfrac{\dfrac{\overset{\phi}{\vdash B, \Delta} \qquad \overset{\tau}{\vdash B, \Delta}}{\vdash B, \Delta}\,(\cup) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \otimes B, \Gamma}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}\,(cut)$$

$$\overset{\not\beta^{+}}{\swarrow} \qquad\qquad\qquad \overset{\not\beta^{+}}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}\,(\cup) \qquad\qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash B, \Delta} \qquad \overset{\tau}{\vdash B, \Delta}}{\vdash B, \Delta}\,(\cup)}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)$$

- $\otimes - \emptyset$ *commutations*

$$\dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A^{\perp}, A \otimes B, \Delta}\,(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}\,(cut)$$

$$\overset{\not\beta^{+}}{\swarrow} \qquad\qquad\qquad \overset{\not\beta^{+}}{\searrow}$$

$$\dfrac{}{\vdash A \otimes B, \Gamma, \Delta}\,(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)$$

$$\dfrac{\dfrac{}{\vdash B,\Delta}(\emptyset) \qquad \dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix} \qquad \dfrac{}{\vdash B^{\perp}, B}(ax)}{\vdash B^{\perp}, A\otimes B,\Gamma}(\otimes)}{\vdash A\otimes B,\Gamma,\Delta}(cut)$$

$$\swarrow{\scriptstyle \overline{\beta}^{\times}} \qquad\qquad \searrow{\scriptstyle \overline{\beta}^{\times}}$$

$$\dfrac{}{\vdash A\otimes B,\Gamma,\Delta}(\emptyset) \qquad\qquad \dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix} \qquad \dfrac{}{\vdash B,\Delta}(\emptyset)}{\vdash A\otimes B,\Gamma,\Delta}(\otimes)$$

- $\bot - \bot$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdash \Gamma\end{matrix}}{\vdash \bot,\Gamma}(\bot)}{\vdash \bot,\bot_2,\Gamma}(\bot_2) \qquad \dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1,\bot_1}(\bot_1)}{\vdash \bot_1,\bot_2,\Gamma}(cut)$$

$$\swarrow{\scriptstyle \overline{\beta}^{\times}} \qquad\qquad \searrow{\scriptstyle \overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdash \Gamma\end{matrix}}{\vdash \bot_1,\Gamma}(\bot_1)}{\vdash \bot_1,\bot_2,\Gamma}(\bot_2) \qquad\qquad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdash \Gamma\end{matrix}}{\vdash \bot_2,\Gamma}(\bot_2)}{\vdash \bot_1,\bot_2,\Gamma}(\bot_1)$$

- $\bot - \bot$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix}}{\vdash A,\bot,\Gamma}(\bot) \qquad \dfrac{\begin{matrix}\phi\\ \vdash B,\Gamma\end{matrix}}{\vdash B,\bot,\Gamma}(\bot)}{\vdash A \,\&\, B,\bot,\Gamma}(\&) \qquad \dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1,\bot}(\bot)}{\vdash A \,\&\, B,\bot,\Gamma}(cut)$$

$$\swarrow{\scriptstyle \overline{\beta}^{\times}} \qquad\qquad \searrow{\scriptstyle \overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix}}{\vdash A,\bot,\Gamma}(\bot) \qquad \dfrac{\begin{matrix}\phi\\ \vdash B,\Gamma\end{matrix}}{\vdash B,\bot,\Gamma}(\bot)}{\vdash A \,\&\, B,\bot,\Gamma}(\&) \qquad\qquad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix} \qquad \begin{matrix}\phi\\ \vdash B,\Gamma\end{matrix}}{\vdash A \,\&\, B,\Gamma}(\&)}{\vdash A \,\&\, B,\bot,\Gamma}(\bot)$$

- $\bot - \oplus_1$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix}}{\vdash A,\bot,\Gamma}(\bot)}{\vdash A\oplus B,\bot,\Gamma}(\oplus_1) \qquad \dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1,\bot}(\bot)}{\vdash A\oplus B,\bot,\Gamma}(cut)$$

$$\swarrow{\scriptstyle \overline{\beta}^{\times}} \qquad\qquad \searrow{\scriptstyle \overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix}}{\vdash A,\bot,\Gamma}(\bot)}{\vdash A\oplus B,\bot,\Gamma}(\oplus_1) \qquad\qquad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdash A,\Gamma\end{matrix}}{\vdash A\oplus B,\Gamma}(\oplus_1)}{\vdash A\oplus B,\bot,\Gamma}(\bot)$$

- $\bot - \oplus_2$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B,\Gamma\end{array}}{\vdash B,\bot,\Gamma}\,(\bot)}{\vdash A\oplus B,\bot,\Gamma}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1,\bot}\,(\bot)}{\vdash A\oplus B,\bot,\Gamma}\,(cut)$$

$$\swarrow\,\overline{\beta}^{\times} \qquad\qquad\qquad\qquad \overline{\beta}^{\times}\,\searrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B,\Gamma\end{array}}{\vdash B,\bot,\Gamma}\,(\bot)}{\vdash A\oplus B,\bot,\Gamma}\,(\oplus_2) \qquad\qquad\qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B,\Gamma\end{array}}{\vdash A\oplus B,\Gamma}\,(\oplus_2)}{\vdash A\oplus B,\bot,\Gamma}\,(\bot)$$

- $\bot - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{}{\vdash \top,\bot,\Gamma}\,(\top) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1,\bot}\,(\bot)}{\vdash \top,\bot,\Gamma}\,(cut)$$

$$\swarrow\,\overline{\beta}^{\times} \qquad\qquad\qquad\qquad \overline{\beta}^{\times}\,\searrow$$

$$\dfrac{}{\vdash \top,\bot,\Gamma}\,(\top) \qquad\qquad\qquad\qquad \dfrac{\dfrac{}{\vdash \top,\Gamma}\,(\top)}{\vdash \top,\bot,\Gamma}\,(\bot)$$

- $\bot - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A,\Gamma\end{array}}{\vdash A,\bot,\Gamma}\,(\bot)}{\vdash ?A,\bot,\Gamma}\,(?_d) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1,\bot}\,(\bot)}{\vdash ?A,\bot,\Gamma}\,(cut)$$

$$\swarrow\,\overline{\beta}^{\times} \qquad\qquad\qquad\qquad \overline{\beta}^{\times}\,\searrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A,\Gamma\end{array}}{\vdash A,\bot,\Gamma}\,(\bot)}{\vdash ?A,\bot,\Gamma}\,(?_d) \qquad\qquad\qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A,\Gamma\end{array}}{\vdash ?A,\Gamma}\,(?_d)}{\vdash ?A,\bot,\Gamma}\,(\bot)$$

- $\bot - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, ?A, \bot, \Gamma}\,(\bot)}{\vdash ?A, \bot, \Gamma}\,(?_c) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash ?A, \bot, \Gamma}\,(cut)$$

$$\overset{\not\beta}{\swarrow} \qquad\qquad\qquad \overset{\not\beta}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, ?A, \bot, \Gamma}\,(\bot)}{\vdash ?A, \bot, \Gamma}\,(?_c) \qquad\qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c)}{\vdash ?A, \bot, \Gamma}\,(\bot)$$

- $\bot - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot)}{\vdash ?A, \bot, \Gamma}\,(?_w) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash ?A, \bot, \Gamma}\,(cut)$$

$$\overset{\not\beta}{\swarrow} \qquad\qquad\qquad \overset{\not\beta}{\searrow}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot)}{\vdash ?A, \bot, \Gamma}\,(?_w) \qquad\qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w)}{\vdash ?A, \bot, \Gamma}\,(\bot)$$

- $\bot - \forall$ *commutation*

$$\dfrac{X \text{ not free in } \bot,\Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \bot, A, \Gamma}\,(\bot)}{\vdash \bot, \forall X A, \Gamma}\,(\forall) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash \bot, \forall X A, \Gamma}\,(cut)$$

$$\overset{\not\beta}{\swarrow} \qquad\qquad\qquad \overset{\not\beta}{\searrow}$$

$$X \text{ not free in } \bot,\Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \bot, A, \Gamma}\,(\bot)}{\vdash \bot, \forall X A, \Gamma}\,(\forall) \qquad\qquad X \text{ not free in } \Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \forall X A, \Gamma}\,(\forall)}{\vdash \bot, \forall X A, \Gamma}\,(\bot)$$

- $\bot - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma}}{\vdash \bot, A[B/X],\Gamma}\,(\bot)}{\vdash \bot, \exists XA, \Gamma}\,(\exists) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash \bot, \exists XA, \Gamma}\,(cut)$$

$$\cancel{\beta}\swarrow \qquad\qquad \searrow\cancel{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma}}{\vdash \bot, A[B/X],\Gamma}\,(\bot)}{\vdash \bot, \exists XA, \Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma}}{\vdash \exists XA,\Gamma}\,(\exists)}{\vdash \bot, \exists XA, \Gamma}\,(\bot)$$

- $\bot - mix_2$ *commutations*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot,\Gamma}\,(\bot) \quad \overset{\phi}{\vdash \Delta}}{\vdash \bot,\Gamma,\Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash \bot,\Gamma,\Delta}\,(cut)$$

$$\cancel{\beta}\swarrow \qquad\qquad \searrow\cancel{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot,\Gamma}\,(\bot) \quad \overset{\phi}{\vdash \Delta}}{\vdash \bot,\Gamma,\Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma,\Delta}\,(mix_2)}{\vdash \bot,\Gamma,\Delta}\,(\bot)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash \bot,\Delta}\,(\bot)}{\vdash \bot,\Gamma,\Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash \bot,\Gamma,\Delta}\,(cut)$$

$$\cancel{\beta}\swarrow \qquad\qquad \searrow\cancel{\beta}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash \bot,\Delta}\,(\bot)}{\vdash \bot,\Gamma,\Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma,\Delta}\,(mix_2)}{\vdash \bot,\Gamma,\Delta}\,(\bot)$$

- $\bot - \cup$ *commutation*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot) \qquad \dfrac{\overset{\phi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot)}{\vdash \bot, \Gamma}\,(\cup) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash \bot, \Gamma}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot) \qquad \dfrac{\overset{\phi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot)}{\vdash \bot, \Gamma}\,(\cup) \qquad\qquad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup)}{\vdash \bot, \Gamma}\,(\bot)$$

- $\bot - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash \bot, \Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, \bot}\,(\bot)}{\vdash \bot, \Gamma}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{}{\vdash \bot, \Gamma}\,(\emptyset) \qquad\qquad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{}{\vdash \Gamma}\,(\emptyset)}{\vdash \bot, \Gamma}\,(\bot)$$

- $\& - \&$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&) \qquad \dfrac{\overset{\tau}{\vdash A, D, \Gamma} \quad \overset{\mu}{\vdash B, D, \Gamma}}{\vdash A \,\&\, B, D, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \,\&\, D, \Gamma}\,(\&) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\bot, A}\,(ax)}{\vdash B \oplus A^\bot, A}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^\bot, B}\,(ax)}{\vdash B \oplus A^\bot, B}\,(\oplus_1)}{\vdash B^\bot \oplus A^\bot, A \,\&\, B}\,(\&)}{\vdash A \,\&\, B, C \,\&\, D, \Gamma}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&) \qquad \dfrac{\overset{\tau}{\vdash A, D, \Gamma} \quad \overset{\mu}{\vdash B, D, \Gamma}}{\vdash A \,\&\, B, D, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \,\&\, D, \Gamma}\,(\&)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\tau}{\vdash A, D, \Gamma}}{\vdash A, C \,\&\, D, \Gamma}\,(\&) \qquad \dfrac{\overset{\phi}{\vdash B, C, \Gamma} \quad \overset{\mu}{\vdash B, D, \Gamma}}{\vdash B, C \,\&\, D, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \,\&\, D, \Gamma}\,(\&)$$

- $\& - \oplus_1$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash C^{\perp}, C \oplus D}\,(\oplus_1) \qquad \dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \oplus D, \Gamma}\,(cut)$$

$$\swarrow \not\beta \qquad\qquad \not\beta \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \oplus D, \Gamma}\,(\oplus_1) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_1) \qquad \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, C \oplus D, \Gamma}\,(\oplus_1)}{\vdash A \,\&\, B, C \oplus D, \Gamma}\,(\&)$$

- $\& - \oplus_2$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash D^{\perp}, D}\,(ax)}{\vdash D^{\perp}, C \oplus D}\,(\oplus_2) \qquad \dfrac{\overset{\pi}{\vdash A, D, \Gamma} \qquad \overset{\phi}{\vdash B, D, \Gamma}}{\vdash A \,\&\, B, D, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \oplus D, \Gamma}\,(cut)$$

$$\swarrow \not\beta \qquad\qquad \not\beta \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma} \qquad \overset{\phi}{\vdash B, D, \Gamma}}{\vdash A \,\&\, B, D, \Gamma}\,(\&)}{\vdash A \,\&\, B, C \oplus D, \Gamma}\,(\oplus_1) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_1) \qquad \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, C \oplus D, \Gamma}\,(\oplus_1)}{\vdash A \,\&\, B, C \oplus D, \Gamma}\,(\&)$$

- $\& - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{}{\vdash \top, 0}\,(\top) \qquad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \qquad \dfrac{}{\vdash B, \top, \Gamma}\,(\top)}{\vdash A \,\&\, B, \top, \Gamma}\,(\&)}{\vdash A \,\&\, B, \top, \Gamma}\,(cut)$$

$$\swarrow \not\beta \qquad\qquad \not\beta \searrow$$

$$\dfrac{}{\vdash A \,\&\, B, \top, \Gamma}\,(\top) \qquad\qquad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \qquad \dfrac{}{\vdash B, \top, \Gamma}\,(\top)}{\vdash A \,\&\, B, \top, \Gamma}\,(\&)$$

- $\& - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(?_d) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash B^{\perp} \oplus A^{\perp}, A}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp} \oplus A^{\perp}, B}\,(\oplus_1)}{\vdash B^{\perp} \oplus A^{\perp}, A \,\&\, B}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(cut)$$

$$\swarrow \not\beta \qquad\qquad \not\beta \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(?_d) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_d) \qquad \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_d)}{\vdash A \,\&\, B, ?C, \Gamma}\,(\&)$$

- $\& - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma} \qquad \overset{\phi}{\vdash B, ?C, ?C, \Gamma}}{\vdash A \,\&\, B, ?C, ?C, \Gamma}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(?_c) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash B^\perp \oplus A^\perp, A}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp \oplus A^\perp, B}\,(\oplus_1)}{\vdash B^\perp \oplus A^\perp, A \,\&\, B}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma} \qquad \overset{\phi}{\vdash B, ?C, ?C, \Gamma}}{\vdash A \,\&\, B, ?C, ?C, \Gamma}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(?_c) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_c) \qquad \dfrac{\overset{\phi}{\vdash B, ?C, ?C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_c)}{\vdash A \,\&\, B, ?C, \Gamma}\,(\&)$$

- $\& - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Gamma}}{\vdash A \,\&\, B, \Gamma}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(?_w) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash B^\perp \oplus A^\perp, A}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp \oplus A^\perp, B}\,(\oplus_1)}{\vdash B^\perp \oplus A^\perp, A \,\&\, B}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Gamma}}{\vdash A \,\&\, B, \Gamma}\,(\&)}{\vdash A \,\&\, B, ?C, \Gamma}\,(?_w) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_w) \qquad \dfrac{\overset{\phi}{\vdash B, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_w)}{\vdash A \,\&\, B, ?C, \Gamma}\,(\&)$$

- $\& - \forall$ *commutation*

$$\dfrac{X \text{ not free in } A\&B,\Gamma \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&)}{\vdash A \,\&\, B, \forall X C, \Gamma}\,(\forall) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash B^\perp \oplus A^\perp, A}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp \oplus A^\perp, B}\,(\oplus_1)}{\vdash B^\perp \oplus A^\perp, A \,\&\, B}\,(\&)}{\vdash A \,\&\, B, \forall X C, \Gamma}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$X \text{ not free in } A\&B,\Gamma \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \,\&\, B, C, \Gamma}\,(\&)}{\vdash A \,\&\, B, \forall X C, \Gamma}\,(\forall)$$

$$\dfrac{X \text{ not free in } A,\Gamma \quad \dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, \forall X C, \Gamma}\,(\forall) \qquad X \text{ not free in } B,\Gamma \quad \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, \forall X C, \Gamma}\,(\forall)}{\vdash A \,\&\, B, \forall X C, \Gamma}\,(\&)$$

- $\& - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma} \qquad \overset{\phi}{\vdash B, C[D/X], \Gamma}}{\vdash A \mathbin{\&} B, C[D/X], \Gamma}\,(\&)}{\vdash A \mathbin{\&} B, \exists X C, \Gamma}\,(\exists) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash B^{\perp} \oplus A^{\perp}, A}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp} \oplus A^{\perp}, B}\,(\oplus_1)}{\vdash B^{\perp} \oplus A^{\perp}, A \mathbin{\&} B}\,(\&)}{\vdash A \mathbin{\&} B, \exists X C, \Gamma}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma} \qquad \overset{\phi}{\vdash B, C[D/X], \Gamma}}{\vdash A \mathbin{\&} B, C[D/X], \Gamma}\,(\&)}{\vdash A \mathbin{\&} B, \exists X C, \Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma}}{\vdash A, \exists X C, \Gamma}\,(\exists) \qquad \dfrac{\overset{\phi}{\vdash B, C[D/X], \Gamma}}{\vdash B, \exists X C, \Gamma}\,(\exists)}{\vdash A \mathbin{\&} B, \exists X C, \Gamma}\,(\&)$$

- $\& - mix_2$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \qquad \overset{\tau}{\vdash B, \Delta}}{\vdash A \mathbin{\&} B, \Delta}\,(\&)}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,(ax)}{\vdash A, B^{\perp} \oplus A^{\perp}}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B, B^{\perp}}\,(ax)}{\vdash B, B^{\perp} \oplus A^{\perp}}\,(\oplus_1)}{\vdash A \mathbin{\&} B, B^{\perp} \oplus A^{\perp}}\,(\&)}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \qquad \overset{\tau}{\vdash B, \Delta}}{\vdash A \mathbin{\&} B, \Delta}\,(\&)}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\tau}{\vdash B, \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(\&)$$

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, \Gamma}}{\vdash A \mathbin{\&} B, \Gamma}\,(\&) \qquad \overset{\phi}{\vdash \Delta}}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,(ax)}{\vdash A, B^{\perp} \oplus A^{\perp}}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B, B^{\perp}}\,(ax)}{\vdash B, B^{\perp} \oplus A^{\perp}}\,(\oplus_1)}{\vdash A \mathbin{\&} B, B^{\perp} \oplus A^{\perp}}\,(\&)}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, \Gamma}}{\vdash A \mathbin{\&} B, \Gamma}\,(\&) \qquad \overset{\phi}{\vdash \Delta}}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\overset{\tau}{\vdash B, \Gamma} \qquad \overset{\phi}{\vdash \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \mathbin{\&} B, \Gamma, \Delta}\,(\&)$$

- $\& - \cup$ *commutation*

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\phi}{\vdash B,\Gamma}}{\vdash A \,\&\, B,\Gamma}\,(\&) \quad \dfrac{\overset{\tau}{\vdash A,\Gamma} \quad \overset{\mu}{\vdash B,\Gamma}}{\vdash A \,\&\, B,\Gamma}\,(\&)}{\vdash A \,\&\, B,\Gamma}\,(\cup) \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax)}{\vdash A, B^\perp \oplus A^\perp}\,(\oplus_2) \quad \dfrac{\dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash B, B^\perp \oplus A^\perp}\,(\oplus_1)}{\vdash A \,\&\, B, B^\perp \oplus A^\perp}\,(\&)
$$
$$
\dfrac{\qquad\qquad\qquad\qquad\qquad\qquad}{\vdash A \,\&\, B,\Gamma,}\,(cut)
$$

$$\swarrow_{\not\beta} \qquad\qquad\qquad\qquad \searrow_{\not\beta}$$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\phi}{\vdash B,\Gamma}}{\vdash A \,\&\, B,\Gamma}\,(\&) \quad \dfrac{\overset{\tau}{\vdash A,\Gamma} \quad \overset{\mu}{\vdash B,\Gamma}}{\vdash A \,\&\, B,\Gamma}\,(\&)}{\vdash A \,\&\, B,\Gamma}\,(\cup)
$$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\tau}{\vdash A,\Gamma}}{\vdash A,\Gamma}\,(\cup) \quad \dfrac{\overset{\phi}{\vdash B,\Gamma} \quad \overset{\mu}{\vdash B,\Gamma}}{\vdash B,\Gamma}\,(\cup)}{\vdash A \,\&\, B,\Gamma}\,(\&)
$$

- $\& - \emptyset$ *commutation*

$$
\dfrac{\dfrac{}{\vdash A \,\&\, B,\Gamma}\,(\emptyset) \quad \dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax)}{\vdash A, B^\perp \oplus A^\perp}\,(\oplus_2) \quad \dfrac{\dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash B, B^\perp \oplus A^\perp}\,(\oplus_1)}{\vdash A \,\&\, B, B^\perp \oplus A^\perp}\,(\&)}{\vdash A \,\&\, B,\Gamma,}\,(cut)
$$

$$\swarrow_{\not\beta} \qquad\qquad\qquad\qquad \searrow_{\not\beta}$$

$$
\dfrac{}{\vdash A \,\&\, B,\Gamma}\,(\emptyset) \qquad\qquad\qquad \dfrac{\dfrac{}{\vdash A,\Gamma}\,(\emptyset) \quad \dfrac{}{\vdash B,\Gamma}\,(\emptyset)}{\vdash A \,\&\, B,\Gamma}\,(\&)
$$

- $\oplus_1 - \oplus_1$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A, C,\Gamma}}{\vdash A, C \oplus D,\Gamma}\,(\oplus_1) \quad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A^\perp, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D,\Gamma}\,(cut)
$$

$$\swarrow_{\not\beta} \qquad\qquad\qquad\qquad \searrow_{\not\beta}$$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash A, C,\Gamma}}{\vdash A \oplus B, C,\Gamma}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D,\Gamma}\,(\oplus_1) \qquad\qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, C,\Gamma}}{\vdash A, C \oplus D,\Gamma}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D,\Gamma}\,(\oplus_1)
$$

- $\oplus_1 - \oplus_2$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\begin{tikzcd}[column sep=small]
 & \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, D, \Gamma\end{array}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_2) \quad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(cut) \arrow[dl, "\overline{\beta}^{+}"'] \arrow[dr, "\overline{\beta}^{+}"] & \\
\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, D, \Gamma\end{array}}{\vdash A \oplus B, D, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_2) & & \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, D, \Gamma\end{array}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_1)
\end{tikzcd}
$$

- $\oplus_1 - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\begin{tikzcd}[column sep=small]
 & \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \quad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, \top, \Gamma}\,(cut) \arrow[dl, "\overline{\beta}^{+}"'] \arrow[dr, "\overline{\beta}^{+}"] & \\
\dfrac{}{\vdash A \oplus B, \top, \Gamma}\,(\top) & & \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top)}{\vdash A \oplus B, \top, \Gamma}\,(\oplus_1)
\end{tikzcd}
$$

- $\oplus_1 - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\begin{tikzcd}[column sep=small]
 & \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, C, \Gamma\end{array}}{\vdash A, ?C, \Gamma}\,(?_d) \quad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(cut) \arrow[dl, "\overline{\beta}^{+}"'] \arrow[dr, "\overline{\beta}^{+}"] & \\
\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, C, \Gamma\end{array}}{\vdash A \oplus B, C, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(?_d) & & \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, C, \Gamma\end{array}}{\vdash A, ?C, \Gamma}\,(?_d)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_1)
\end{tikzcd}
$$

- $\oplus_1 - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\begin{tikzcd}[column sep=small]
 & \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, ?C, ?C, \Gamma\end{array}}{\vdash A, ?C, \Gamma}\,(?_c) \quad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(cut) \arrow[dl, "\overline{\beta}^{+}"'] \arrow[dr, "\overline{\beta}^{+}"] & \\
\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, ?C, ?C, \Gamma\end{array}}{\vdash A \oplus B, ?C, ?C, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(?_c) & & \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A, ?C, ?C, \Gamma\end{array}}{\vdash A, ?C, \Gamma}\,(?_c)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_1)
\end{tikzcd}
$$

- $\oplus_1 - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,?C,\Gamma}\,(?_w) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A^\perp, A\oplus B}\,(\oplus_1)}{\vdash A\oplus B, ?C,\Gamma}\,(cut)$$

$$\swarrow \overrightarrow{\beta}^{\times} \qquad\qquad \overrightarrow{\beta}^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_1)}{\vdash A\oplus B, ?C,\Gamma}\,(?_w) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,?C,\Gamma}\,(?_w)}{\vdash A\oplus B, ?C,\Gamma}\,(\oplus_1)$$

- $\oplus_1 - \forall$ *commutation*

$$\dfrac{X \text{ not free in } A,\Gamma\ \dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,\forall XC,\Gamma}\,(\forall) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A^\perp, A\oplus B}\,(\oplus_1)}{\vdash A\oplus B, \forall XC,\Gamma}\,(cut)$$

$$\swarrow \overrightarrow{\beta}^{\times} \qquad\qquad \overrightarrow{\beta}^{\times} \searrow$$

$$X \text{ not free in } A\oplus B,\Gamma\ \dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A\oplus B,C,\Gamma}\,(\oplus_1)}{\vdash A\oplus B, \forall XC,\Gamma}\,(\forall) \qquad\qquad \dfrac{X \text{ not free in } A,\Gamma\ \dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,\forall XC,\Gamma}\,(\forall)}{\vdash A\oplus B, \forall XC,\Gamma}\,(\oplus_1)$$

- $\oplus_1 - \exists$ *commutation*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,C[D/X],\Gamma}}{\vdash A,\exists XC,\Gamma}\,(\exists) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A^\perp, A\oplus B}\,(\oplus_1)}{\vdash A\oplus B, \exists XC,\Gamma}\,(cut)$$

$$\swarrow \overrightarrow{\beta}^{\times} \qquad\qquad \overrightarrow{\beta}^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,C[D/X],\Gamma}}{\vdash A\oplus B,C[D/X],\Gamma}\,(\oplus_1)}{\vdash A\oplus B, \exists XC,\Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A,C[D/X],\Gamma}}{\vdash A,\exists XC,\Gamma}\,(\exists)}{\vdash A\oplus B, \exists XC,\Gamma}\,(\oplus_1)$$

- $\oplus_1 - mix_2$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A,\Gamma,\Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A^\perp, A\oplus B}\,(\oplus_1)}{\vdash A\oplus B, \Gamma,\Delta}\,(cut)$$

$$\swarrow \overrightarrow{\beta}^{\times} \qquad\qquad \overrightarrow{\beta}^{\times} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_1) \quad \overset{\phi}{\vdash \Delta}}{\vdash A\oplus B, \Gamma,\Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A,\Gamma,\Delta}\,(mix_2)}{\vdash A\oplus B, \Gamma,\Delta}\,(\oplus_1)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, \Gamma, \Delta}\,(cut)$$

$$\overline{\beta}^{\times}\swarrow \qquad\qquad \searrow\overline{\beta}^{\times}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{\overset{\phi}{\vdash A, \Delta}}{\vdash A \oplus B, \Delta}\,(\oplus_1)}{\vdash A \oplus B, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(\oplus_1)$$

- $\oplus_1 - \cup$ *commutation*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, \Gamma}\,(cut)$$

$$\overline{\beta}^{\times}\swarrow \qquad\qquad \searrow\overline{\beta}^{\times}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_1) \qquad \dfrac{\overset{\phi}{\vdash A, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, \Gamma}\,(\cup) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup)}{\vdash A \oplus B, \Gamma}\,(\oplus_1)$$

- $\oplus_1 - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, A \oplus B}\,(\oplus_1)}{\vdash A \oplus B, \Gamma}\,(cut)$$

$$\overline{\beta}^{\times}\swarrow \qquad\qquad \searrow\overline{\beta}^{\times}$$

$$\dfrac{}{\vdash A \oplus B, \Gamma}\,(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset)}{\vdash A \oplus B, \Gamma}\,(\oplus_1)$$

- $\oplus_2 - \oplus_2$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash B, D, \Gamma}}{\vdash B, C \oplus D, \Gamma}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(cut)$$

$$\overline{\beta}^{\times}\swarrow \qquad\qquad \searrow\overline{\beta}^{\times}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B, D, \Gamma}}{\vdash A \oplus B, D, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash B, D, \Gamma}}{\vdash B, C \oplus D, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_2)$$

- $\oplus_2 - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{}{\vdash B, \top, \Gamma}\,(\top) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, \top, \Gamma}\,(cut)$$

$$\dfrac{}{\vdash A \oplus B, \top, \Gamma}\,(\top) \quad \xleftarrow{\not\beta} \quad \cdot \quad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{}{\vdash B, \top, \Gamma}\,(\top)}{\vdash A \oplus B, \top, \Gamma}\,(\oplus_2)$$

- $\oplus_2 - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash B, C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_d) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, ?C, \Gamma}\,(cut)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B, C, \Gamma}}{\vdash A \oplus B, C, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, ?C, \Gamma}\,(?_d) \quad \xleftarrow{\not\beta} \quad \cdot \quad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B, C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_d)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_2)$$

- $\oplus_2 - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash B, ?C, ?C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_c) \qquad \dfrac{\dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash B^\perp, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, ?C, \Gamma}\,(cut)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B, ?C, ?C, \Gamma}}{\vdash A \oplus B, ?C, ?C, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, ?C, \Gamma}\,(?_c) \quad \xleftarrow{\not\beta} \quad \cdot \quad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B, ?C, ?C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_c)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_2)$$

- $\oplus_2 - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma}}{\vdash B,?C,\Gamma}\,(?_w) \qquad \dfrac{\dfrac{}{\vdash B^{\perp},B}\,(ax)}{\vdash B^{\perp},A\oplus B}\,(\oplus_2)}{\vdash A\oplus B,?C,\Gamma}\,(cut)$$

$$\swarrow \not\to_{\beta} \qquad\qquad \not\to_{\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_2)}{\vdash A\oplus B,?C,\Gamma}\,(?_w) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma}}{\vdash B,?C,\Gamma}\,(?_w)}{\vdash A\oplus B,?C,\Gamma}\,(\oplus_2)$$

- $\oplus_2 - \forall$ *commutation*

$$\dfrac{X \text{ not free in } B,\Gamma\;\; \dfrac{\overset{\pi}{\vdash B,C,\Gamma}}{\vdash B,\forall XC,\Gamma}\,(\forall) \qquad \dfrac{\dfrac{}{\vdash B^{\perp},B}\,(ax)}{\vdash B^{\perp},A\oplus B}\,(\oplus_2)}{\vdash A\oplus B,\forall XC,\Gamma}\,(cut)$$

$$\swarrow \not\to_{\beta} \qquad\qquad \not\to_{\beta} \searrow$$

$$X \text{ not free in } A\oplus B,\Gamma\;\; \dfrac{\dfrac{\overset{\pi}{\vdash B,C,\Gamma}}{\vdash A\oplus B,C,\Gamma}\,(\oplus_2)}{\vdash A\oplus B,\forall XC,\Gamma}\,(\forall) \qquad\qquad \dfrac{X \text{ not free in } B,\Gamma\;\; \dfrac{\overset{\pi}{\vdash B,C,\Gamma}}{\vdash B,\forall XC,\Gamma}\,(\forall)}{\vdash A\oplus B,\forall XC,\Gamma}\,(\oplus_2)$$

- $\oplus_2 - \exists$ *commutation*

$$\dfrac{\dfrac{\overset{\pi}{\vdash B,C[D/X],\Gamma}}{\vdash B,\exists XC,\Gamma}\,(\exists) \qquad \dfrac{\dfrac{}{\vdash B^{\perp},B}\,(ax)}{\vdash B^{\perp},A\oplus B}\,(\oplus_2)}{\vdash A\oplus B,\exists XC,\Gamma}\,(cut)$$

$$\swarrow \not\to_{\beta} \qquad\qquad \not\to_{\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B,C[D/X],\Gamma}}{\vdash A\oplus B,C[D/X],\Gamma}\,(\oplus_2)}{\vdash A\oplus B,\exists XC,\Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash B,C[D/X],\Gamma}}{\vdash B,\exists XC,\Gamma}\,(\exists)}{\vdash A\oplus B,\exists XC,\Gamma}\,(\oplus_2)$$

- $\oplus_2 - mix_2$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash B,\Gamma,\Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash B^{\perp},B}\,(ax)}{\vdash B^{\perp},A\oplus B}\,(\oplus_2)}{\vdash A\oplus B,\Gamma,\Delta}\,(cut)$$

$$\swarrow \not\to_{\beta} \qquad\qquad \not\to_{\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_2) \qquad \overset{\phi}{\vdash \Delta}}{\vdash A\oplus B,\Gamma,\Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash B,\Gamma,\Delta}\,(mix_2)}{\vdash A\oplus B,\Gamma,\Delta}\,(\oplus_2)$$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{\overset{\pi}{\vdash B, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash A \oplus B, \Delta}\,(\oplus_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(mix_2)
\qquad\qquad
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(\oplus_2)
$$

- $\oplus_2 - \cup$ *commutation*

$$
\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma} \qquad \overset{\phi}{\vdash B, \Gamma}}{\vdash B, \Gamma}\,(\cup) \qquad \dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, \Gamma}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2) \qquad \dfrac{\overset{\phi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, \Gamma}\,(\cup)
\qquad\qquad
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma} \qquad \overset{\phi}{\vdash B, \Gamma}}{\vdash B, \Gamma}\,(\cup)}{\vdash A \oplus B, \Gamma}\,(\oplus_2)
$$

- $\oplus_2 - \emptyset$ *commutation*

$$
\dfrac{\dfrac{}{\vdash B, \Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash B^{\perp}, A \oplus B}\,(\oplus_2)}{\vdash A \oplus B, \Gamma}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{}{\vdash A \oplus B, \Gamma}\,(\emptyset)
\qquad\qquad
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{\dfrac{}{\vdash B, \Gamma}\,(\emptyset)}{\vdash A \oplus B, \Gamma}\,(\oplus_2)
$$

- $\top - \top$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\dfrac{\dfrac{}{\vdash \top, \top_2, \Gamma}\,(\top_2) \qquad \dfrac{}{\vdash \top_1, 0}\,(\top_1)}{\vdash \top_1, \top_2, \Gamma}\,(cut)
$$

$$
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{}{\vdash \top_1, \top_2, \Gamma}\,(\top_2)
\qquad\qquad
\xrightarrow{\overline{\beta}^{+}} \quad \dfrac{}{\vdash \top_1, \top_2, \Gamma}\,(\top_1)
$$

- $\top - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash \top, A, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_d) \qquad \dfrac{}{\vdash \top, 0}(\top)}{\vdash \top, ?A, \Gamma}(cut)$$

$$\dfrac{\dfrac{}{\vdash \top, A, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_d) \quad \overset{\overline{\beta}^{+}}{\swarrow} \qquad\qquad \overset{\overline{\beta}^{+}}{\searrow} \quad \dfrac{}{\vdash \top, ?A, \Gamma}(\top)$$

- $\top - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_c) \qquad \dfrac{}{\vdash \top, 0}(\top)}{\vdash \top, ?A, \Gamma}(cut)$$

$$\dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_c) \quad \overset{\overline{\beta}^{+}}{\swarrow} \qquad\qquad \overset{\overline{\beta}^{+}}{\searrow} \quad \dfrac{}{\vdash \top, ?A, \Gamma}(\top)$$

- $\top - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{}{\vdash \top, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_w) \qquad \dfrac{}{\vdash \top, 0}(\top)}{\vdash \top, ?A, \Gamma}(cut)$$

$$\dfrac{\dfrac{}{\vdash \top, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_w) \quad \overset{\overline{\beta}^{+}}{\swarrow} \qquad\qquad \overset{\overline{\beta}^{+}}{\searrow} \quad \dfrac{}{\vdash \top, ?A, \Gamma}(\top)$$

- $\top - \forall$ *commutation*

$$\dfrac{X \text{ not free in } \top{,}\Gamma \; \dfrac{\dfrac{}{\vdash \top, A, \Gamma}(\top)}{\vdash \top, \forall X A, \Gamma}(\forall) \qquad \dfrac{}{\vdash \top, 0}(\top)}{\vdash \top, \forall X A, \Gamma}(cut)$$

$$X \text{ not free in } \top{,}\Gamma \; \dfrac{\dfrac{}{\vdash \top, A, \Gamma}(\top)}{\vdash \top, \forall X A, \Gamma}(\forall) \quad \overset{\overline{\beta}^{+}}{\swarrow} \qquad\qquad \overset{\overline{\beta}^{+}}{\searrow} \quad \dfrac{}{\vdash \top, \forall X A, \Gamma}(\top)$$

- $\top - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{}{\vdash \top, A[B/X], \Gamma}(\top)}{\vdash \top, \exists X A, \Gamma}(\exists) \qquad \dfrac{}{\vdash \top, 0}(\top)}{\vdash \top, \exists X A, \Gamma}(cut)$$

$$\dfrac{\dfrac{}{\vdash \top, A[B/X], \Gamma}(\top)}{\vdash \top, \exists X A, \Gamma}(\exists) \quad \overset{\overline{\beta}^{+}}{\swarrow} \qquad\qquad \overset{\overline{\beta}^{+}}{\searrow} \quad \dfrac{}{\vdash \top, \exists X A, \Gamma}(\top)$$

- $\top - mix_2$ *commutations*

$$\dfrac{\dfrac{\dfrac{}{\vdash \top,\Gamma}(\top) \qquad \overset{\pi}{\vdash \Delta}}{\vdash \top,\Gamma,\Delta}(mix_2) \qquad \dfrac{}{\vdash 0,\top}(\top)}{\vdash \top,\Gamma,\Delta}(cut)$$

$$\xrightarrow{\beta^+} \quad \dfrac{\dfrac{}{\vdash \top,\Gamma}(\top) \qquad \overset{\pi}{\vdash \Delta}}{\vdash \top,\Gamma,\Delta}(mix_2) \qquad\qquad \xrightarrow{\beta^+} \quad \dfrac{}{\vdash \top,\Gamma,\Delta}(\top)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash \top,\Delta}(\top)}{\vdash \top,\Gamma,\Delta}(mix_2) \qquad \dfrac{}{\vdash 0,\top}(\top)}{\vdash \top,\Gamma,\Delta}(cut)$$

$$\xrightarrow{\beta^+} \quad \dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash \top,\Delta}(\top)}{\vdash \top,\Gamma,\Delta}(mix_2) \qquad\qquad \xrightarrow{\beta^+} \quad \dfrac{}{\vdash \top,\Gamma,\Delta}(\top)$$

- $\top - \cup$ *commutation*

$$\dfrac{\dfrac{\dfrac{}{\vdash \top,\Gamma}(\top) \qquad \dfrac{}{\vdash \top,\Gamma}(\top)}{\vdash \top,\Gamma}(\cup) \qquad \dfrac{}{\vdash \top,0}(\top)}{\vdash \top,\Gamma}(cut)$$

$$\xrightarrow{\beta^+} \quad \dfrac{\dfrac{}{\vdash \top,\Gamma}(\top) \qquad \dfrac{}{\vdash \top,\Gamma}(\top)}{\vdash \top,\Gamma}(\cup) \qquad\qquad \xrightarrow{\beta^+} \quad \dfrac{}{\vdash \top,\Gamma}(\top)$$

- $\top - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash \top,\Gamma}(\emptyset) \qquad \dfrac{}{\vdash \top,0}(\top)}{\vdash \top,\Gamma}(cut)$$

$$\xrightarrow{\beta^+} \quad \dfrac{}{\vdash \top,\Gamma}(\emptyset) \qquad\qquad \xrightarrow{\beta^+} \quad \dfrac{}{\vdash \top,\Gamma}(\top)$$

- $?_d - ?_d$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A,B,\Gamma}}{\vdash A,?B,\Gamma}(?_d) \qquad \dfrac{\dfrac{}{\vdash A^\perp,A}(ax)}{\vdash A^\perp,?A}(?_d)}{\vdash ?A,?B,\Gamma}(cut)$$

$$\xrightarrow{\beta^+} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A,B,\Gamma}}{\vdash ?A,B,\Gamma}(?_d)}{\vdash ?A,?B,\Gamma}(?_d) \qquad\qquad \xrightarrow{\beta^+} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A,B,\Gamma}}{\vdash A,?B,\Gamma}(?_d)}{\vdash ?A,?B,\Gamma}(?_d)$$

- $?_d - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_c) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(cut)$$

$$\swarrow {\scriptstyle \not\beta} \qquad\qquad\qquad {\scriptstyle \not\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, \Gamma}}{\vdash ?A, ?B, ?B, \Gamma}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(?_c) \qquad\qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_c)}{\vdash ?A, ?B, \Gamma}\,(?_d)$$

- $?_d - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_w) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(cut)$$

$$\swarrow {\scriptstyle \not\beta} \qquad\qquad\qquad {\scriptstyle \not\beta} \searrow$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(?_w) \qquad\qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_d)$$

- $?_d - \forall$ *commutation*

$$\dfrac{X \text{ not free in } A,\Gamma\ \dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, \forall X B, \Gamma}\,(\forall) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, \forall X B, \Gamma}\,(cut)$$

$$\swarrow {\scriptstyle \not\beta} \qquad\qquad\qquad {\scriptstyle \not\beta} \searrow$$

$$X \text{ not free in } ?A,\Gamma\ \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash ?A, B, \Gamma}\,(?_d)}{\vdash ?A, \forall X B, \Gamma}\,(\forall) \qquad\qquad \dfrac{X \text{ not free in } A,\Gamma\ \dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, \forall X B, \Gamma}\,(\forall)}{\vdash ?A, \forall X B, \Gamma}\,(?_d)$$

- $?_d - \exists$ *commutation*

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash A, B[C/X], \Gamma}}{\vdash A, \exists X B, \Gamma}\,(\exists) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, \exists X B, \Gamma}\,(cut)
$$

$$
\swarrow_{\overline{\beta}^{+}} \qquad\qquad \searrow^{\overline{\beta}^{+}}
$$

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash A, B[C/X], \Gamma}}{\vdash A, B[C/X], \Gamma}\,(?_d)}{\vdash ?A, \exists X B, \Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\stackrel{\pi}{\vdash A, B[C/X], \Gamma}}{\vdash A, \exists X B, \Gamma}\,(\exists)}{\vdash ?A, \exists X B, \Gamma}\,(?_d)
$$

- $?_d - mix_2$ *commutations*

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash A, \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, \Gamma, \Delta}\,(cut)
$$

$$
\swarrow_{\overline{\beta}^{+}} \qquad\qquad \searrow^{\overline{\beta}^{+}}
$$

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d) \quad \stackrel{\phi}{\vdash \Delta}}{\vdash ?A, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\stackrel{\pi}{\vdash A, \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_d)
$$

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, \Gamma, \Delta}\,(cut)
$$

$$
\swarrow_{\overline{\beta}^{+}} \qquad\qquad \searrow^{\overline{\beta}^{+}}
$$

$$
\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \dfrac{\stackrel{\phi}{\vdash A, \Delta}}{\vdash ?A, \Delta}\,(?_d)}{\vdash ?A, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_d)
$$

- $?_d - \cup$ *commutation*

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash A, \Gamma} \quad \stackrel{\phi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash A^{\perp}, ?A}\,(?_d)}{\vdash ?A, \Gamma}\,(cut)
$$

$$
\swarrow_{\overline{\beta}^{+}} \qquad\qquad \searrow^{\overline{\beta}^{+}}
$$

$$
\dfrac{\dfrac{\stackrel{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d) \quad \dfrac{\stackrel{\phi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d)}{\vdash ?A, \Gamma}\,(\cup) \qquad\qquad \dfrac{\dfrac{\stackrel{\pi}{\vdash A, \Gamma} \quad \stackrel{\phi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup)}{\vdash ?A, \Gamma}\,(?_d)
$$

- $?_d - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash A,\Gamma}(\emptyset) \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}(ax)}{\vdash A^{\perp}, ?A}(?_d)}{\vdash ?A,\Gamma}(cut)$$

$$\swarrow^{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{}{\vdash ?A,\Gamma}(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash A,\Gamma}(\emptyset)}{\vdash ?A,\Gamma}(?_d)$$

- $?_c - ?_c$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, ?B, ?B, \Gamma}}{\vdash ?A \parr ?A, ?B, ?B, \Gamma}(\parr)}{\vdash ?A \parr ?A, ?B, \Gamma}(?_c) \qquad \dfrac{\dfrac{\dfrac{}{\vdash !A, ?A}(ax) \quad \dfrac{}{\vdash !A, ?A}(ax)}{\vdash !A \otimes !A, ?A, ?A}(\otimes)}{\vdash !A \otimes !A, ?A}(?_c)}{\vdash ?A, ?B, \Gamma}(cut)$$

$$\swarrow^{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, ?B, ?B, \Gamma}}{\vdash ?A, ?B, ?B, \Gamma}(?_c)}{\vdash ?A, ?B, \Gamma}(?_c) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, ?B, ?B, \Gamma}}{\vdash ?A, ?A, ?B, \Gamma}(?_c)}{\vdash ?A, ?B, \Gamma}(?_c)$$

- $?_c - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A \parr ?A, \Gamma}(\parr)}{\vdash ?A \parr ?A, ?B, \Gamma}(?_w) \qquad \dfrac{\dfrac{\dfrac{}{\vdash !A, ?A}(ax) \quad \dfrac{}{\vdash !A, ?A}(ax)}{\vdash !A \otimes !A, ?A, ?A}(\otimes)}{\vdash !A \otimes !A, ?A}(?_c)}{\vdash ?A, ?B, \Gamma}(cut)$$

$$\swarrow^{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}(?_c)}{\vdash ?A, ?B, \Gamma}(?_w) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, ?A, ?B, \Gamma}(?_w)}{\vdash ?A, ?B, \Gamma}(?_c)$$

- $?_c - \forall$ *commutation*

$$\dfrac{X \text{ not free in } \exists X B^{\perp} \; \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}(ax)}{\vdash \exists X B^{\perp}, B}(\exists)}{\vdash \exists X B^{\perp}, \forall X B}(\forall) \qquad X \text{ not free in } ?A,?A,\Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B, \Gamma}}{\vdash ?A, ?A, \forall X B, \Gamma}(\forall)}{\vdash ?A, \forall X B, \Gamma}(?_c)}{\vdash ?A, \forall X B, \Gamma}(cut)$$

$$\swarrow^{\overline{\beta}^{\times}} \qquad\qquad \searrow^{\overline{\beta}^{\times}}$$

$$X \text{ not free in } ?A,\Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B, \Gamma}}{\vdash ?A, B, \Gamma}(?_c)}{\vdash ?A, \forall X B, \Gamma}(\forall) \qquad\qquad X \text{ not free in } ?A,?A,\Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B, \Gamma}}{\vdash ?A, ?A, \forall X B, \Gamma}(\forall)}{\vdash ?A, \forall X B, \Gamma}(?_c)$$

- $?_c - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{}{\vdash B[C/X], (B[C/X])^{\perp}}\,(ax)}{\vdash \exists X B, (B[C/X])^{\perp}}\,(\exists) \qquad \dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, B[C/X], \Gamma}\,(?_c)}{\vdash ?A, \exists X B, \Gamma}\,(cut)$$

$$\swarrow \beta \qquad\qquad \searrow \beta$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, ?A, B[C/X], \Gamma}\,(?_c)}{\vdash ?A, \exists X B, \Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, ?A, \exists X B, \Gamma}\,(\exists)}{\vdash ?A, \exists X B, \Gamma}\,(?_c)$$

- $?_c - mix_2$ *commutations*

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A \parr ?A, \Gamma}\,(\parr) \qquad \overset{\phi}{\vdash \Delta}}{\vdash ?A \parr ?A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{\dfrac{}{\vdash !A, ?A}\,(ax) \qquad \dfrac{}{\vdash !A, ?A}\,(ax)}{\vdash !A \otimes !A, ?A, ?A}\,(\otimes)}{\vdash !A \otimes !A, ?A}\,(?_c)}{\vdash ?A, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta \qquad\qquad \searrow \beta$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c) \qquad \overset{\phi}{\vdash \Delta}}{\vdash ?A, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma} \qquad \overset{\phi}{\vdash \Delta}}{\vdash ?A, ?A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_c)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{\overset{\phi}{\vdash ?A, ?A, \Delta}}{\vdash ?A \parr ?A, \Delta}\,(\parr)}{\vdash ?A \parr ?A, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{\dfrac{}{\vdash !A, ?A}\,(ax) \qquad \dfrac{}{\vdash !A, ?A}\,(ax)}{\vdash !A \otimes !A, ?A, ?A}\,(\otimes)}{\vdash !A \otimes !A, ?A}\,(?_c)}{\vdash ?A, \Gamma, \Delta}\,(cut)$$

$$\swarrow \beta \qquad\qquad \searrow \beta$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{\overset{\phi}{\vdash ?A, ?A, \Delta}}{\vdash ?A, \Delta}\,(?_c)}{\vdash ?A, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash ?A, ?A, \Delta}}{\vdash ?A, ?A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_c)$$

- $?_c - \cup$ *commutation*

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A \parr ?A, \Gamma}\,(\parr) \qquad \dfrac{\dfrac{\phi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A \parr ?A, \Gamma}\,(\parr)}{\vdash ?A \parr ?A, \Gamma}\,(\cup) \qquad \dfrac{\dfrac{\dfrac{}{\vdash !A, ?A}\,(ax) \qquad \dfrac{}{\vdash !A, ?A}\,(ax)}{\vdash !A \otimes !A, ?A, ?A}\,(\otimes)}{\vdash !A \otimes !A, ?A}\,(?_c)
$$

$$
\dfrac{\vdash ?A \parr ?A, \Gamma \qquad \vdash !A \otimes !A, ?A}{\vdash ?A, \Gamma}\,(cut)
$$

$$
\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c) \qquad \dfrac{\dfrac{\phi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c)}{\vdash ?A, \Gamma}\,(\cup) \qquad\qquad \dfrac{\dfrac{\dfrac{\pi}{\vdash ?A, ?A, \Gamma} \qquad \dfrac{\phi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(\cup)}{\vdash ?A, \Gamma}\,(?_c)
$$

- $?_c - \emptyset$ *commutation*

$$
\dfrac{\dfrac{}{\vdash ?A \parr ?A, \Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{\dfrac{}{\vdash !A, ?A}\,(ax) \qquad \dfrac{}{\vdash !A, ?A}\,(ax)}{\vdash !A \otimes !A, ?A, ?A}\,(\otimes)}{\vdash !A \otimes !A, ?A}\,(?_c)}{\vdash ?A, \Gamma}\,(cut)
$$

$$
\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow
$$

$$
\dfrac{}{\vdash ?A, \Gamma}\,(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash ?A, ?A, \Gamma}\,(\emptyset)}{\vdash ?A, \Gamma}\,(?_c)
$$

- $?_w - ?_w$ *commutation* (See this cut-elimination in C1ick ⅋ c⊗LLec⊥ here.)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot)}{\vdash \bot, ?B, \Gamma}\,(?_w) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, ?A}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(cut)
$$

$$
\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow
$$

$$
\dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_w) \qquad\qquad \dfrac{\dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash ?B, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_w)
$$

- $?_w - \forall$ *commutation*

$$\dfrac{X \text{ not free in } \exists XB^{\perp} \quad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,(ax)}{\vdash \exists XB^{\perp}, B}\,(\exists)}{\vdash \exists XB^{\perp}, \forall XB}\,(\forall) \qquad X \text{ not free in } \Gamma \quad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B, \Gamma\end{array}}{\vdash \forall XB, \Gamma}\,(\forall)}{\vdash ?A, \forall XB, \Gamma}\,(?_w)}{\vdash ?A, \forall XB, \Gamma}\,(cut)$$

$$\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow$$

$$X \text{ not free in } ?A{,}\Gamma \quad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B, \Gamma\end{array}}{\vdash ?A, B, \Gamma}\,(?_w)}{\vdash ?A, \forall XB, \Gamma}\,(\forall) \qquad\qquad X \text{ not free in } \Gamma \quad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B, \Gamma\end{array}}{\vdash \forall XB, \Gamma}\,(\forall)}{\vdash ?A, \forall XB, \Gamma}\,(?_w)$$

- $?_w - \exists$ *commutation*

$$\dfrac{\dfrac{\dfrac{}{\vdash B[C/X], (B[C/X])^{\perp}}\,(ax)}{\vdash \exists XB, (B[C/X])^{\perp}}\,(\exists) \qquad \dfrac{\begin{array}{c}\pi\\ \vdash B[C/X], \Gamma\end{array}}{\vdash ?A, B[C/X], \Gamma}\,(?_w)}{\vdash ?A, \exists XB, \Gamma}\,(cut)$$

$$\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B[C/X], \Gamma\end{array}}{\vdash B[C/X], \Gamma}\,(?_w)}{\vdash ?A, \exists XB, \Gamma}\,(\exists) \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash B[C/X], \Gamma\end{array}}{\vdash \exists XB, \Gamma}\,(\exists)}{\vdash ?A, \exists XB, \Gamma}\,(?_w)$$

- $?_w - mix_2$ *commutations*

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash \Gamma\end{array}}{\vdash \bot, \Gamma}\,(\bot) \quad \begin{array}{c}\phi\\ \vdash \Delta\end{array}}{\vdash \bot, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, ?A}\,(?_w)}{\vdash ?A, \Gamma, \Delta}\,(cut)$$

$$\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash \Gamma\end{array}}{\vdash ?A, \Gamma}\,(?_w) \quad \begin{array}{c}\phi\\ \vdash \Delta\end{array}}{\vdash ?A, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash \Gamma\end{array} \quad \begin{array}{c}\phi\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_w)$$

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash \Gamma\end{array} \quad \dfrac{\begin{array}{c}\phi\\ \vdash \Delta\end{array}}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash 1, ?A}\,(?_w)}{\vdash ?A, \Gamma, \Delta}\,(cut)$$

$$\swarrow \overline{\beta}^{\times} \qquad\qquad \overline{\beta}^{\times} \searrow$$

$$\dfrac{\begin{array}{c}\pi\\ \vdash \Gamma\end{array} \quad \dfrac{\begin{array}{c}\phi\\ \vdash \Delta\end{array}}{\vdash ?A, \Delta}\,(?_w)}{\vdash ?A, \Gamma, \Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash \Gamma\end{array} \quad \begin{array}{c}\phi\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_w)$$

- $?_w - \cup$ *commutation*

$$\cfrac{\cfrac{\cfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot,\Gamma}(\bot) \qquad \cfrac{\overset{\phi}{\vdash \Gamma}}{\vdash \bot,\Gamma}(\bot)}{\vdash \bot,\Gamma}(\cup) \qquad \cfrac{\cfrac{}{\vdash 1}(1)}{\vdash 1, ?A}(?_w)}{\vdash ?A,\Gamma}(cut)$$

$$\swarrow \cancel{\beta}^{+} \qquad\qquad \cancel{\beta}^{+} \searrow$$

$$\cfrac{\cfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A,\Gamma}(?_w) \qquad \cfrac{\overset{\phi}{\vdash \Gamma}}{\vdash ?A,\Gamma}(?_w)}{\vdash ?A,\Gamma}(\cup) \qquad\qquad \cfrac{\cfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Gamma}}{\vdash \Gamma}(\cup)}{\vdash ?A,\Gamma}(?_w)$$

- $?_w - \emptyset$ *commutation*

$$\cfrac{\cfrac{}{\vdash \bot,\Gamma}(\emptyset) \qquad \cfrac{\cfrac{}{\vdash 1}(1)}{\vdash 1, ?A}(?_w)}{\vdash ?A,\Gamma}(cut)$$

$$\swarrow \cancel{\beta}^{+} \qquad\qquad \cancel{\beta}^{+} \searrow$$

$$\cfrac{}{\vdash ?A,\Gamma}(\emptyset) \qquad\qquad \cfrac{\cfrac{}{\vdash \Gamma}(\emptyset)}{\vdash ?A,\Gamma}(?_w)$$

- $\forall - \forall$ *commutation*

$$\cfrac{{\scriptstyle Y \text{ not free in } \forall XA,\Gamma}\ \cfrac{{\scriptstyle X \text{ not free in } B,\Gamma}\ \cfrac{\overset{\pi}{\vdash A,B,\Gamma}}{\vdash \forall XA,B,\Gamma}(\forall)}{\vdash \forall XA,\forall YB,\Gamma}(\forall) \qquad {\scriptstyle X \text{ not free in } \exists XA^{\perp}}\ \cfrac{\cfrac{\cfrac{}{\vdash A^{\perp},A}(ax)}{\vdash \exists XA^{\perp},A}(\exists)}{\vdash \exists XA^{\perp},\forall XA}(\forall)}{\vdash \forall XA,\forall YB,\Gamma}(cut)$$

$$\swarrow \cancel{\beta}^{+} \qquad\qquad \cancel{\beta}^{+} \searrow$$

$${\scriptstyle Y \text{ not free in } \forall XA,\Gamma}\ \cfrac{{\scriptstyle X \text{ not free in } B,\Gamma}\ \cfrac{\overset{\pi}{\vdash A,B,\Gamma}}{\vdash \forall XA,B,\Gamma}(\forall)}{\vdash \forall XA,\forall YB,\Gamma}(\forall) \qquad\qquad {\scriptstyle X \text{ not free in } \forall YB,\Gamma}\ \cfrac{{\scriptstyle Y \text{ not free in } A,\Gamma}\ \cfrac{\overset{\pi}{\vdash A,B,\Gamma}}{\vdash A,\forall YB,\Gamma}(\forall)}{\vdash \forall XA,\forall YB,\Gamma}(\forall)$$

- $\forall - \exists$ *commutation*

$$
\dfrac{X \text{ not free in } B[C/Y],\Gamma \;\; \dfrac{\dfrac{\overset{\pi}{\vdash A, B[C/Y], \Gamma}}{\vdash \forall XA, B[C/Y], \Gamma}\,(\forall)}{\vdash \forall XA, \exists YB, \Gamma}\,(\exists) \qquad X \text{ not free in } \exists XA^{\perp} \;\; \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash \exists XA^{\perp}, A}\,(\exists)}{\vdash \exists XA^{\perp}, \forall XA}\,(\forall)}{\vdash \forall XA, \exists YB, \Gamma}\,(cut)
$$

$$
\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow
$$

$$
X \text{ not free in } B[C/Y],\Gamma \;\; \dfrac{\dfrac{\overset{\pi}{\vdash A, B[C/Y], \Gamma}}{\vdash \forall XA, B[C/Y], \Gamma}\,(\forall)}{\vdash \forall XA, \exists YB, \Gamma}\,(\exists) \qquad\qquad X \text{ not free in } \exists YB,\Gamma \;\; \dfrac{\dfrac{\overset{\pi}{\vdash A, B[C/Y], \Gamma}}{\vdash A, \exists YB, \Gamma}\,(\exists)}{\vdash \forall XA, \exists YB, \Gamma}\,(\forall)
$$

- $\forall - mix_2$ *commutations*

$$
\dfrac{\dfrac{X \text{ not free in } \Gamma \;\; \dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \forall XA, \Gamma}\,(\forall) \quad \overset{\phi}{\vdash \Delta}}{\vdash \forall XA, \Gamma, \Delta}\,(mix_2) \qquad X \text{ not free in } \exists XA^{\perp} \;\; \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash \exists XA^{\perp}, A}\,(\exists)}{\vdash \exists XA^{\perp}, \forall XA}\,(\forall)}{\vdash \forall XA, \Gamma, \Delta}\,(cut)
$$

$$
\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow
$$

$$
\dfrac{X \text{ not free in } \Gamma \;\; \dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \forall XA, \Gamma}\,(\forall) \quad \overset{\phi}{\vdash \Delta}}{\vdash \forall XA, \Gamma, \Delta}\,(mix_2) \qquad\qquad X \text{ not free in } \Gamma,\Delta \;\; \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash \forall XA, \Gamma, \Delta}\,(\forall)
$$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad X \text{ not free in } \Delta \;\; \dfrac{\overset{\phi}{\vdash A, \Delta}}{\vdash \forall XA, \Delta}\,(\forall)}{\vdash \forall XA, \Gamma, \Delta}\,(mix_2) \qquad X \text{ not free in } \exists XA^{\perp} \;\; \dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash \exists XA^{\perp}, A}\,(\exists)}{\vdash \exists XA^{\perp}, \forall XA}\,(\forall)}{\vdash \forall XA, \Gamma, \Delta}\,(cut)
$$

$$
\swarrow \beta^{\times} \qquad\qquad \beta^{\times} \searrow
$$

$$
\dfrac{\overset{\pi}{\vdash \Gamma} \quad X \text{ not free in } \Delta \;\; \dfrac{\overset{\phi}{\vdash A, \Delta}}{\vdash \forall XA, \Delta}\,(\forall)}{\vdash \forall XA, \Gamma, \Delta}\,(mix_2) \qquad\qquad X \text{ not free in } \Gamma,\Delta \;\; \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash \forall XA, \Gamma, \Delta}\,(\forall)
$$

- $\forall - \cup$ *commutation*

$$\dfrac{\dfrac{X\text{ not free in }\Gamma\;\dfrac{\begin{array}{c}\pi\\ \vdash A,\Gamma\end{array}}{\vdash \forall XA,\Gamma}\,(\forall)\qquad X\text{ not free in }\Gamma\;\dfrac{\begin{array}{c}\phi\\ \vdash A,\Gamma\end{array}}{\vdash \forall XA,\Gamma}\,(\forall)}{\vdash \forall XA,\Gamma}\,(\cup)\qquad X\text{ not free in }\exists XA^{\perp}\;\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp},A}\,(ax)}{\vdash \exists XA^{\perp},A}\,(\exists)}{\vdash \exists XA^{\perp},\forall XA}\,(\forall)}{\vdash \forall XA,\Gamma}\,(cut)$$

$$\swarrow\overline{\beta}^{\times}\qquad\qquad\overline{\beta}^{\times}\searrow$$

$$\dfrac{X\text{ not free in }\Gamma\;\dfrac{\begin{array}{c}\pi\\ \vdash A,\Gamma\end{array}}{\vdash \forall XA,\Gamma}\,(\forall)\qquad X\text{ not free in }\Gamma\;\dfrac{\begin{array}{c}\phi\\ \vdash A,\Gamma\end{array}}{\vdash \forall XA,\Gamma}\,(\forall)}{\vdash \forall XA,\Gamma}\,(\cup)\qquad\qquad X\text{ not free in }\Gamma\;\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A,\Gamma\end{array}\quad\begin{array}{c}\phi\\ \vdash A,\Gamma\end{array}}{\vdash A,\Gamma}\,(\cup)}{\vdash \forall XA,\Gamma}\,(\forall)$$

- $\forall - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash \forall XA,\Gamma}\,(\emptyset)\qquad X\text{ not free in }\exists XA^{\perp}\;\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp},A}\,(ax)}{\vdash \exists XA^{\perp},A}\,(\exists)}{\vdash \exists XA^{\perp},\forall XA}\,(\forall)}{\vdash \forall XA,\Gamma}\,(cut)$$

$$\swarrow\overline{\beta}^{\times}\qquad\qquad\overline{\beta}^{\times}\searrow$$

$$\dfrac{}{\vdash \forall XA,\Gamma}\,(\emptyset)\qquad\qquad X\text{ not free in }\Gamma\;\dfrac{\dfrac{}{\vdash A,\Gamma}\,(\emptyset)}{\vdash \forall XA,\Gamma}\,(\forall)$$

- $\exists - \exists$ *commutation*

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A[C/X],B[D/Y],\Gamma\end{array}}{\vdash A[C/X],B[D/Y],\Gamma}\,(\exists)\qquad\dfrac{\dfrac{}{\vdash A[C/X],(A[C/X])^{\perp}}\,(ax)}{\vdash \exists XA,(A[C/X])^{\perp}}\,(\exists)}{\vdash \exists XA,\exists YB,\Gamma}\,(cut)$$

$$\swarrow\overline{\beta}^{\times}\qquad\qquad\overline{\beta}^{\times}\searrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A[C/X],B[D/Y],\Gamma\end{array}}{\vdash \exists XA,B[D/Y],\Gamma}\,(\exists)}{\vdash \exists XA,\exists YB,\Gamma}\,(\exists)\qquad\qquad\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdash A[C/X],B[D/Y],\Gamma\end{array}}{\vdash A[C/X],\exists YB,\Gamma}\,(\exists)}{\vdash \exists XA,\exists YB,\Gamma}\,(\exists)$$

- $\exists - mix_2$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A[B/X],\Gamma,\Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A[B/X],(A[B/X])^{\perp}}\,(ax)}{\vdash \exists XA,(A[B/X])^{\perp}}\,(\exists)}{\vdash \exists XA,\Gamma,\Delta}\,(cut)$$

$$\overrightarrow{\beta}^{\times}\swarrow \qquad \searrow\overrightarrow{\beta}^{\times}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma}}{\vdash \exists XA,\Gamma}\,(\exists) \quad \overset{\phi}{\vdash \Delta}}{\vdash \exists XA,\Gamma,\Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A[B/X],\Gamma,\Delta}\,(mix_2)}{\vdash \exists XA,\Gamma,\Delta}\,(\exists)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A[B/X],\Delta}}{\vdash A[B/X],\Gamma,\Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash A[B/X],(A[B/X])^{\perp}}\,(ax)}{\vdash \exists XA,(A[B/X])^{\perp}}\,(\exists)}{\vdash \exists XA,\Gamma,\Delta}\,(cut)$$

$$\overrightarrow{\beta}^{\times}\swarrow \qquad \searrow\overrightarrow{\beta}^{\times}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A[B/X],\Delta}}{\vdash \exists XA,\Delta}\,(\exists)}{\vdash \exists XA,\Gamma,\Delta}\,(mix_2) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A[B/X],\Delta}}{\vdash A[B/X],\Gamma,\Delta}\,(mix_2)}{\vdash \exists XA,\Gamma,\Delta}\,(\exists)$$

- $\exists - \cup$ *commutation*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma} \quad \overset{\phi}{\vdash A[B/X],\Gamma}}{\vdash A[B/X],\Gamma}\,(\cup) \qquad \dfrac{\dfrac{}{\vdash A[B/X],(A[B/X])^{\perp}}\,(ax)}{\vdash \exists XA,(A[B/X])^{\perp}}\,(\exists)}{\vdash \exists XA,\Gamma}\,(cut)$$

$$\overrightarrow{\beta}^{\times}\swarrow \qquad \searrow\overrightarrow{\beta}^{\times}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma}}{\vdash \exists XA,\Gamma}\,(\exists) \quad \dfrac{\overset{\phi}{\vdash A[B/X],\Gamma}}{\vdash \exists XA,\Gamma}\,(\exists)}{\vdash \exists XA,\Gamma}\,(\cup) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash A[B/X],\Gamma} \quad \overset{\phi}{\vdash A[B/X],\Gamma}}{\vdash A[B/X],\Gamma}\,(\cup)}{\vdash \exists XA,\Gamma}\,(\exists)$$

- $\exists - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash A[B/X],\Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{}{\vdash A[B/X],(A[B/X])^{\perp}}\,(ax)}{\vdash \exists XA,(A[B/X])^{\perp}}\,(\exists)}{\vdash \exists XA,\Gamma}\,(cut)$$

$$\overrightarrow{\beta}^{\times}\swarrow \qquad \searrow\overrightarrow{\beta}^{\times}$$

$$\dfrac{}{\vdash \exists XA,\Gamma}\,(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash A[B/X],\Gamma}\,(\emptyset)}{\vdash \exists XA,\Gamma}\,(\exists)$$

- $mix_2 - mix_2$ *commutations*

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot, \Gamma, \Delta}\,(mix_2) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1) \quad \overset{\tau}{\vdash \Sigma}}{\vdash 1, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta} \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Delta, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$$

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \overset{\tau}{\vdash \Sigma}}{\vdash 1, \Sigma}\,(mix_2) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot) \quad \overset{\phi}{\vdash \Delta}}{\vdash \bot, \Gamma, \Delta}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Sigma}\,(mix_2) \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1, \Gamma}\,(mix_2) \qquad \dfrac{\overset{\phi}{\vdash \Delta} \quad \dfrac{\overset{\tau}{\vdash \Sigma}}{\vdash \bot, \Sigma}\,(\bot)}{\vdash \bot, \Delta, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta} \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Delta, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2) \qquad\qquad \xrightarrow{\not\beta} \quad \dfrac{\overset{\phi}{\vdash \Delta} \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$$

- $mix_2 - \cup$ *commutations*

$$\dfrac{\dfrac{\dfrac{\overset{\phi}{\vdash \Delta}}{\vdash \bot, \Delta}\,(\bot) \quad \dfrac{\overset{\tau}{\vdash \Delta}}{\vdash \bot, \Delta}\,(\bot)}{\vdash \bot, \Delta}\,(\cup) \qquad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1, \Gamma}\,(mix_2)}{\vdash \Gamma, \Delta}\,(cut)$$

$$\xrightarrow{\not\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash \Gamma, \Delta}\,(\cup) \qquad\qquad \xrightarrow{\not\beta} \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta} \quad \overset{\tau}{\vdash \Delta}}{\vdash \Delta}\,(\cup)}{\vdash \Gamma, \Delta}\,(mix_2)$$

$$
\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot) \quad \dfrac{\overset{\tau}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot)}{\vdash \bot, \Gamma}(\cup) \qquad \dfrac{\dfrac{}{\vdash 1}(1) \quad \overset{\phi}{\vdash \Delta}}{\vdash 1, \Delta}(mix_2)}{\vdash \Gamma, \Delta}(cut)
$$

$\swarrow \overline{\beta}^{+}$ $\qquad$ $\overline{\beta}^{+} \searrow$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}(mix_2) \quad \dfrac{\overset{\tau}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}(mix_2)}{\vdash \Gamma, \Delta}(\cup) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash \Gamma}}{\vdash \Gamma}(\cup) \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}(mix_2)
$$

- *$mix_2 - \emptyset$ commutations*

$$
\dfrac{\dfrac{}{\vdash \bot, \Gamma}(\emptyset) \qquad \dfrac{\dfrac{}{\vdash 1}(1) \quad \overset{\pi}{\vdash \Delta}}{\vdash 1, \Delta}(mix_2)}{\vdash \Gamma, \Delta}(cut)
$$

$\swarrow \overline{\beta}^{+}$ $\qquad$ $\overline{\beta}^{+} \searrow$

$$
\dfrac{}{\vdash \Gamma, \Delta}(\emptyset) \qquad\qquad \dfrac{\dfrac{}{\vdash \Gamma}(\emptyset) \quad \overset{\pi}{\vdash \Delta}}{\vdash \Gamma, \Delta}(mix_2)
$$

$$
\dfrac{\dfrac{}{\vdash \bot, \Delta}(\emptyset) \qquad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{}{\vdash 1}(1)}{\vdash 1, \Gamma}(mix_2)}{\vdash \Gamma, \Delta}(cut)
$$

$\swarrow \overline{\beta}^{+}$ $\qquad$ $\overline{\beta}^{+} \searrow$

$$
\dfrac{}{\vdash \Gamma, \Delta}(\emptyset) \qquad\qquad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{}{\vdash \Delta}(\emptyset)}{\vdash \Gamma, \Delta}(mix_2)
$$

- $\cup - \cup$ *commutation*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \Gamma, \bot}(\bot) \quad \dfrac{\overset{\phi}{\vdash \Gamma}}{\vdash \Gamma, \bot}(\bot)}{\vdash \Gamma, \bot \,\&\, \bot}(\&) \quad \dfrac{\dfrac{\overset{\tau}{\vdash \Gamma}}{\vdash \Gamma, \bot}(\bot) \quad \dfrac{\overset{\mu}{\vdash \Gamma}}{\vdash \Gamma, \bot}(\bot)}{\vdash \Gamma, \bot \,\&\, \bot}(\&)}{\vdash \Gamma, \bot \,\&\, \bot}(\cup) \qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1 \oplus 1}(\oplus_2) \quad \dfrac{\dfrac{}{\vdash 1}(1)}{\vdash 1 \oplus 1}(\oplus_1)}{\vdash 1 \oplus 1}(\cup)}{\vdash \Gamma}(cut)
$$

$\swarrow \overline{\beta}^{+}$ $\qquad$ $\overline{\beta}^{+} \searrow$

$$
\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Gamma}}{\vdash \Gamma}(\cup) \quad \dfrac{\overset{\tau}{\vdash \Gamma} \quad \overset{\mu}{\vdash \Gamma}}{\vdash \Gamma}(\cup)}{\vdash \Gamma}(\cup) \qquad\qquad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash \Gamma}}{\vdash \Gamma}(\cup) \quad \dfrac{\overset{\phi}{\vdash \Gamma} \quad \overset{\mu}{\vdash \Gamma}}{\vdash \Gamma}(\cup)}{\vdash \Gamma}(\cup)
$$

- $\cup - \emptyset$ *commutation*

$$\dfrac{\dfrac{}{\vdash \bot, \Gamma}\,(\emptyset) \qquad \dfrac{\dfrac{}{\vdash 1}\,(\emptyset) \qquad \dfrac{}{\vdash 1}\,(\emptyset)}{\vdash 1}\,(\cup)}{\vdash \Gamma}\,(cut)$$

$$\swarrow_{\overline{\beta}^{+}} \qquad\qquad \searrow^{\overline{\beta}^{+}}$$

$$\dfrac{}{\vdash \Gamma}\,(\emptyset) \qquad\qquad\qquad \dfrac{\dfrac{}{\vdash \Gamma}\,(\emptyset) \qquad \dfrac{}{\vdash \Gamma}\,(\emptyset)}{\vdash \Gamma}\,(\cup)$$

$\square$

E Tables defining transformations of derivations

The succeeding pages contain the tables defining the transformations of derivations from Section 1.2.

ax	$\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\pi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(cut)$	$\xrightarrow{\beta}$	$\overset{\pi}{\vdash A, \Gamma}$
$\parr - \otimes - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, A^{\perp}, \Gamma}}{\vdash B^{\perp} \parr A^{\perp}, \Gamma}\,(\parr) \qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, A^{\perp}, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, \Gamma, \Sigma}\,(cut) \qquad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$
$\parr - \otimes - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, A^{\perp}, \Gamma}}{\vdash B^{\perp} \parr A^{\perp}, \Gamma}\,(\parr) \qquad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B^{\perp}, \Gamma, \Delta}\,(cut) \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$
$\perp - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \perp, \Gamma}\,(\perp) \qquad \dfrac{}{\vdash 1}\,(1)}{\vdash \Gamma}\,(cut)$	$\xrightarrow{\beta}$	$\overset{\pi}{\vdash \Gamma}$
$\& - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A^{\perp}, \Gamma}}{\vdash B^{\perp} \,\&\, A^{\perp}, \Gamma}\,(\&) \qquad \dfrac{\overset{\tau}{\vdash A, \Delta}}{\vdash A \oplus B, \Delta}\,(\oplus_1)}{\vdash \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\phi}{\vdash A^{\perp}, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)$
$\& - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A^{\perp}, \Gamma}}{\vdash B^{\perp} \,\&\, A^{\perp}, \Gamma}\,(\&) \qquad \dfrac{\overset{\tau}{\vdash B, \Delta}}{\vdash A \oplus B, \Delta}\,(\oplus_2)}{\vdash \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash B^{\perp}, \Gamma} \quad \overset{\tau}{\vdash B, \Delta}}{\vdash \Gamma, \Delta}\,(cut)$
$?_d - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash ?A^{\perp}, \Gamma}\,(?_d) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, ?\Delta}}{\vdash \Gamma, ?\Delta}\,(cut)$
$?_c - ! - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1^{\perp}, ?A_2^{\perp}, \Gamma}}{\vdash ?A_1^{\perp}, \Gamma}\,(?_c) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1^{\perp}, ?A_2^{\perp}, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash ?A_1^{\perp}, \Gamma, ?\Delta}\,(cut) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta, ?\Delta}\,(cut)}{\vdash \Gamma, ?\Delta}\,(?_c)$
$?_c - ! - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1^{\perp}, ?A_2^{\perp}, \Gamma}}{\vdash ?A_1^{\perp}, \Gamma}\,(?_c) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1^{\perp}, ?A_2^{\perp}, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash ?A_2^{\perp}, \Gamma, ?\Delta}\,(cut) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta, ?\Delta}\,(cut)}{\vdash \Gamma, ?\Delta}\,(?_c)$
$?_w - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A^{\perp}, \Gamma}\,(?_w) \qquad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \Gamma, ?\Delta}\,(?_w)$
$\forall - \exists$	X not free in Γ $\quad \dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash \forall X A^{\perp}, \Gamma}\,(\forall) \qquad \dfrac{\overset{\phi}{\vdash A[B/X], \Delta}}{\vdash \exists X A, \Delta}\,(\exists)}{\vdash \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi[B/X]}{\vdash A[B/X]^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A[B/X], \Delta}}{\vdash \Gamma, \Delta}\,(cut)$

Table 2: Cut-elimination $\xrightarrow{\beta}$ – Key cases

$cut - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B^{\perp}, \Gamma, \Delta}\,(\mathit{cut}) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B^{\perp}, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, \Gamma, \Sigma}\,(\mathit{cut}) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(\mathit{cut})$
$\parr - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, C, \Gamma}}{\vdash A^{\perp}, B \parr C, \Gamma}\,(\parr) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B \parr C, \Gamma, \Delta}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, C, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, C, \Gamma, \Delta}\,(\mathit{cut})}{\vdash B \parr C, \Gamma, \Delta}\,(\parr)$
$\otimes - cut - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\phi}{\vdash C, \Delta}}{\vdash A^{\perp}, B \otimes C, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B, \Gamma, \Sigma}\,(\mathit{cut}) \quad \overset{\phi}{\vdash C, \Delta}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\otimes)$
$\otimes - cut - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \overset{\phi}{\vdash A^{\perp}, C, \Delta}}{\vdash A^{\perp}, B \otimes C, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A^{\perp}, C, \Delta} \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash C, \Delta, \Sigma}\,(\mathit{cut})}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\otimes)$
$\bot - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash A^{\perp}, \bot, \Gamma}\,(\bot) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \bot, \Gamma, \Delta}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(\mathit{cut})}{\vdash \bot, \Gamma, \Delta}\,(\bot)$
$\& - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\phi}{\vdash A^{\perp}, C, \Gamma}}{\vdash A^{\perp}, B \,\&\, C, \Gamma}\,(\&) \quad \overset{\tau}{\vdash A, \Delta}}{\vdash B \,\&\, C, \Gamma, \Delta}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(\mathit{cut}) \quad \dfrac{\overset{\phi}{\vdash A^{\perp}, C, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash C, \Gamma, \Delta}\,(\mathit{cut})}{\vdash B \,\&\, C, \Gamma, \Delta}\,(\&)$
$\oplus_1 - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma}}{\vdash A^{\perp}, B \oplus C, \Gamma}\,(\oplus_1) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B \oplus C, \Gamma, \Delta}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(\mathit{cut})}{\vdash B \oplus C, \Gamma, \Delta}\,(\oplus_1)$
$\oplus_2 - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, C, \Gamma}}{\vdash A^{\perp}, B \oplus C, \Gamma}\,(\oplus_2) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B \oplus C, \Gamma, \Delta}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, C, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash C, \Gamma, \Delta}\,(\mathit{cut})}{\vdash B \oplus C, \Gamma, \Delta}\,(\oplus_2)$
$\top - cut$	$\dfrac{\dfrac{}{\vdash A^{\perp}, \top, \Gamma}\,(\top) \quad \overset{\pi}{\vdash A, \Delta}}{\vdash \top, \Gamma, \Delta}\,(\mathit{cut})$	$\xrightarrow{\beta}$	$\dfrac{}{\vdash \top, \Gamma, \Delta}\,(\top)$

Table 3: Cut-elimination $\xrightarrow{\beta}$ – Commutative cases (Part 1/2)

$?_d - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma}}{\vdash A^{\perp}, ?B, \Gamma}\,(?_d) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?_d)$
$?_c - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, ?B_1, ?B_2, \Gamma}}{\vdash A^{\perp}, ?B_1, \Gamma}\,(?_c) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B_1, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, ?B_1, ?B_2, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B_1, ?B_2, \Gamma, \Delta}\,(cut)}{\vdash ?B_1, \Gamma, \Delta}\,(?_c)$
$?_w - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma}}{\vdash A^{\perp}, ?B, \Gamma}\,(?_w) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?_w)$
$! - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp}, B, ?\Gamma}}{\vdash ?A^{\perp}, !B, ?\Gamma}\,(!) \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash !B, ?\Gamma, ?\Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp}, B, ?\Gamma} \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash B, ?\Gamma, ?\Delta}\,(cut)}{\vdash !B, ?\Gamma, ?\Delta}\,(!)$
$\forall - cut$	$\dfrac{X \text{ not free in } A^{\perp},\Gamma\ \dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma}}{\vdash A^{\perp}, \forall XB, \Gamma}\,(\forall) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \forall XB, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$X \text{ not free in } \Gamma,\Delta\ \dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash \forall XB, \Gamma, \Delta}\,(\forall)$
$\exists - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B[C/X], \Gamma}}{\vdash A^{\perp}, \exists XB, \Gamma}\,(\exists) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \exists XB, \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B[C/X], \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B[C/X]\Gamma, \Delta}\,(cut)}{\vdash \exists XB, \Gamma, \Delta}\,(\exists)$
$mix_2 - cut - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A^{\perp}, \Gamma, \Delta}\,(mix_2) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Sigma}\,(cut) \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$
$mix_2 - cut - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A^{\perp}, \Delta}}{\vdash A^{\perp}, \Gamma, \Delta}\,(mix_2) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A^{\perp}, \Delta} \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash \Delta, \Sigma}\,(cut)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$
$\cup - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\phi}{\vdash A^{\perp}, \Gamma}}{\vdash A^{\perp}, \Gamma}\,(\cup) \quad \overset{\tau}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut) \quad \dfrac{\overset{\phi}{\vdash A^{\perp}, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \Gamma, \Delta}\,(\cup)$
$\emptyset - cut$	$\dfrac{\dfrac{}{\vdash A^{\perp}, \Gamma}\,(\emptyset) \quad \overset{\pi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)$	$\xrightarrow{\beta}$	$\dfrac{}{\vdash \Gamma, \Delta}\,(\emptyset)$

Table 4: Cut-elimination $\xrightarrow{\beta}$ – Commutative cases (Part 2/2)

$\parr - \parr$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, D, \Gamma}}{\vdash A \parr B, C, D, \Gamma}(\parr)}{\vdash A \parr B, C \parr D, \Gamma}(\parr)$	$\xrightarrow{C^{\parr}_{\parr}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, D, \Gamma}}{\vdash A, B, C \parr D, \Gamma}(\parr)}{\vdash A \parr B, C \parr D, \Gamma}(\parr)$
$\parr - \otimes - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}(\parr) \quad \overset{\phi}{\vdash D, \Delta}}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}(\otimes)$	$\xrightarrow{C^{\parr}_{\otimes}}$ $\xleftarrow[C^{\otimes}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma} \quad \overset{\phi}{\vdash D, \Delta}}{\vdash A, B, C \otimes D, \Gamma, \Delta}(\otimes)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}(\parr)$
$\parr - \otimes - 2$	$\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, B, D, \Delta}}{\vdash A \parr B, D, \Delta}(\parr)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}(\otimes)$	$\xrightarrow{C^{\parr}_{\otimes}}$ $\xleftarrow[C^{\otimes}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \overset{\phi}{\vdash A, B, D, \Delta}}{\vdash A, B, C \otimes D, \Gamma, \Delta}(\otimes)}{\vdash A \parr B, C \otimes D, \Gamma, \Delta}(\parr)$
$\parr - \bot$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}(\parr)}{\vdash A \parr B, \bot, \Gamma}(\bot)$	$\xrightarrow{C^{\parr}_{\bot}}$ $\xleftarrow[C^{\bot}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, B, \bot, \Gamma}(\bot)}{\vdash A \parr B, \bot, \Gamma}(\parr)$
$\parr - \&$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}(\parr) \quad \dfrac{\overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}(\parr)}{\vdash A \parr B, C \,\&\, D, \Gamma}(\&)$	$\xrightarrow{C^{\parr}_{\&}}$ $\xleftarrow[C^{\&}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma} \quad \overset{\phi}{\vdash A, B, D, \Gamma}}{\vdash A, B, C \,\&\, D, \Gamma}(\&)}{\vdash A \parr B, C \,\&\, D, \Gamma}(\parr)$
$\parr - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}(\parr)}{\vdash A \parr B, C \oplus D, \Gamma}(\oplus_1)$	$\xrightarrow{C^{\parr}_{\oplus_1}}$ $\xleftarrow[C^{\oplus_1}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A, B, C \oplus D, \Gamma}(\oplus_1)}{\vdash A \parr B, C \oplus D, \Gamma}(\parr)$
$\parr - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, D, \Gamma}}{\vdash A \parr B, D, \Gamma}(\parr)}{\vdash A \parr B, C \oplus D, \Gamma}(\oplus_2)$	$\xrightarrow{C^{\parr}_{\oplus_2}}$ $\xleftarrow[C^{\oplus_2}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, D, \Gamma}}{\vdash A, B, C \oplus D, \Gamma}(\oplus_2)}{\vdash A \parr B, C \oplus D, \Gamma}(\parr)$
$\parr - \top$	$\dfrac{}{\vdash A \parr B, \top, \Gamma}(\top)$	$\xrightarrow{C^{\parr}_{\top}}$ $\xleftarrow[C^{\top}_{\parr}]{}$	$\dfrac{\dfrac{}{\vdash A, B, \top, \Gamma}(\top)}{\vdash A \parr B, \top, \Gamma}(\parr)$
$\parr - ?_d$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A \parr B, C, \Gamma}(\parr)}{\vdash A \parr B, ?C, \Gamma}(?_d)$	$\xrightarrow{C^{\parr}_{?_d}}$ $\xleftarrow[C^{?_d}_{\parr}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C, \Gamma}}{\vdash A, B, ?C, \Gamma}(?_d)}{\vdash A \parr B, ?C, \Gamma}(\parr)$

Table 5: Rule commutation $\overset{r}{\mapsto}$ (Part 1/16)

$\parr - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, ?C, ?C, \Gamma}}{\vdash A \parr B, ?C, ?C, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_c) \quad \underset{C^{?_c}_{\parr}}{\overset{C^{\parr}_{?_c}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, ?C, ?C, \Gamma}}{\vdash A, B, ?C, \Gamma}\,(?_c)}{\vdash A \parr B, ?C, \Gamma}\,(\parr)$
$\parr - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr)}{\vdash A \parr B, ?C, \Gamma}\,(?_w) \quad \underset{C^{?_w}_{\parr}}{\overset{C^{\parr}_{?_w}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, B, ?C, \Gamma}\,(?_w)}{\vdash A \parr B, ?C, \Gamma}\,(\parr)$
$\parr - \forall$	$X \text{ not free in } A \parr B, \Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, C, \Gamma}\,(\parr)}{\vdash A \parr B, \forall X C, \Gamma}\,(\forall) \quad \underset{C^{\forall}_{\parr}}{\overset{C^{\parr}_{\forall}}{\rightleftarrows}} \quad X \text{ not free in } A, B, \Gamma \; \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, B, \forall X C, \Gamma}\,(\forall)}{\vdash A \parr B, \forall X C, \Gamma}\,(\parr)$
$\parr - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, C[D/X], \Gamma}}{\vdash A \parr B, C[D/X], \Gamma}\,(\parr)}{\vdash A \parr B, \exists X C, \Gamma}\,(\exists) \quad \underset{C^{\exists}_{\parr}}{\overset{C^{\parr}_{\exists}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, C[D/X], \Gamma}}{\vdash A, B, \exists X C, \Gamma}\,(\exists)}{\vdash A \parr B, \exists X C, \Gamma}\,(\parr)$
$\parr - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr) \quad \overset{\phi}{\vdash \Delta}}{\vdash A \parr B, \Gamma, \Delta}\,(mix_2) \quad \underset{C^{mix_2}_{\parr}}{\overset{C^{\parr}_{mix_2}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A, B, \Gamma, \Delta}\,(mix_2)}{\vdash A \parr B, \Gamma, \Delta}\,(\parr)$
$\parr - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, B, \Delta}}{\vdash A \parr B, \Delta}\,(\parr)}{\vdash A \parr B, \Gamma, \Delta}\,(mix_2) \quad \underset{C^{mix_2}_{\parr}}{\overset{C^{\parr}_{mix_2}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, B, \Delta}}{\vdash A, B, \Gamma, \Delta}\,(mix_2)}{\vdash A \parr B, \Gamma, \Delta}\,(\parr)$
$\parr - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr) \quad \dfrac{\overset{\phi}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\,(\parr)}{\vdash A \parr B, \Gamma}\,(\cup) \quad \underset{C^{\cup}_{\parr}}{\overset{C^{\parr}_{\cup}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma} \quad \overset{\phi}{\vdash A, B, \Gamma}}{\vdash A, B, \Gamma}\,(\cup)}{\vdash A \parr B, \Gamma}\,(\parr)$
$\parr - \emptyset$	$\dfrac{}{\vdash A \parr B, \Gamma}\,(\emptyset) \quad \underset{C^{\emptyset}_{\parr}}{\overset{C^{\parr}_{\emptyset}}{\rightleftarrows}} \quad \dfrac{\dfrac{}{\vdash A, B, \Gamma}\,(\emptyset)}{\vdash A \parr B, \Gamma}\,(\parr)$
$\otimes - \otimes - 1$	$\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, D, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes) \quad \underset{C^{\otimes}_{\otimes}}{\overset{C^{\otimes}_{\otimes}}{\rightleftarrows}} \quad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \overset{\phi}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$

Table 6: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 2/16)

$\otimes - \otimes - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash D, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$	$\xrightarrow{C^{\otimes}_{\otimes}}$ $\xleftarrow[C^{\otimes}_{\otimes}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\tau}{\vdash D, \Sigma}}{\vdash A, C \otimes D, \Gamma, \Sigma}\,(\otimes) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$
$\otimes - \otimes - 3$	$\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, D, \Sigma}}{\vdash A \otimes B, D, \Gamma, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$	$\xrightarrow{C^{\otimes}_{\otimes}}$ $\xleftarrow[C^{\otimes}_{\otimes}]{}$	$\dfrac{\overset{\phi}{\vdash A, \Delta} \quad \dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \overset{\tau}{\vdash B, D, \Sigma}}{\vdash B, C \otimes D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$
$\otimes - \bot - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \bot, \Gamma, \Delta}\,(\bot)$	$\xrightarrow{C^{\otimes}_{\bot}}$ $\xleftarrow[C^{\bot}_{\otimes}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, \bot, \Gamma}\,(\bot) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \bot, \Gamma, \Delta}\,(\otimes)$
$\otimes - \bot - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \bot, \Gamma, \Delta}\,(\bot)$	$\xrightarrow{C^{\otimes}_{\bot}}$ $\xleftarrow[C^{\bot}_{\otimes}]{}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, \Delta}}{\vdash B, \bot, \Delta}\,(\bot)}{\vdash A \otimes B, \bot, \Gamma, \Delta}\,(\otimes)$
$\otimes - \& - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \quad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&)$	$\xrightarrow{C^{\otimes}_{\&}}$ $\xleftarrow[C^{\&}_{\otimes}]{}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, C, \Delta} \quad \overset{\tau}{\vdash B, D, \Delta}}{\vdash B, C \,\&\, D, \Delta}\,(\&)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes)$
$\otimes - \& - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \quad \dfrac{\overset{\tau}{\vdash A, D, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&)$	$\xrightarrow{C^{\otimes}_{\&}}$ $\xleftarrow[C^{\&}_{\otimes}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\tau}{\vdash A, D, \Gamma}}{\vdash A, C \,\&\, D, \Gamma}\,(\&) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes)$
$\otimes - \oplus_1 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_1)$	$\xrightarrow{C^{\otimes}_{\oplus_1}}$ $\xleftarrow[C^{\oplus_1}_{\otimes}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_1) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$
$\otimes - \oplus_1 - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_1)$	$\xrightarrow{C^{\otimes}_{\oplus_1}}$ $\xleftarrow[C^{\oplus_1}_{\otimes}]{}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, C \oplus D, \Delta}\,(\oplus_1)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$
$\otimes - \oplus_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_2)$	$\xrightarrow{C^{\otimes}_{\oplus_2}}$ $\xleftarrow[C^{\oplus_2}_{\otimes}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_2) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$
$\otimes - \oplus_2 - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\oplus_2)$	$\xrightarrow{C^{\otimes}_{\oplus_2}}$ $\xleftarrow[C^{\oplus_2}_{\otimes}]{}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, D, \Delta}}{\vdash B, C \oplus D, \Delta}\,(\oplus_2)}{\vdash A \otimes B, C \oplus D, \Gamma, \Delta}\,(\otimes)$

Table 7: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 3/16)

$\otimes - \top - 1$	$\dfrac{}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\top)$	$\overset{C^{\otimes}_{\top}}{\longrightarrow}\;\underset{C^{\top}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash B, \Delta}}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\otimes)$
$\otimes - \top - 2$	$\dfrac{}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\top)$	$\overset{C^{\otimes}_{\top}}{\longrightarrow}\;\underset{C^{\top}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B, \top, \Delta}\,(\top)}{\vdash A \otimes B, \top, \Gamma, \Delta}\,(\otimes)$
$\otimes - ?_d - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_d)$	$\overset{C^{\otimes}_{?_d}}{\longrightarrow}\;\underset{C^{?_d}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_d) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$
$\otimes - ?_d - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_d)$	$\overset{C^{\otimes}_{?_d}}{\longrightarrow}\;\underset{C^{?_d}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, ?C, \Delta}\,(?_d)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$
$\otimes - ?_c - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, ?C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_c)$	$\overset{C^{\otimes}_{?_c}}{\longrightarrow}\;\underset{C^{?_c}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_c) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$
$\otimes - ?_c - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, ?C, ?C, \Delta}}{\vdash A \otimes B, ?C, ?C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_c)$	$\overset{C^{\otimes}_{?_c}}{\longrightarrow}\;\underset{C^{?_c}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash B, ?C, ?C, \Delta}}{\vdash B, ?C, \Delta}\,(?_c)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$
$\otimes - ?_w - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_w)$	$\overset{C^{\otimes}_{?_w}}{\longrightarrow}\;\underset{C^{?_w}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_w) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$
$\otimes - ?_w - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(?_w)$	$\overset{C^{\otimes}_{?_w}}{\longrightarrow}\;\underset{C^{?_w}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\phi}{\vdash B, \Delta}}{\vdash B, ?C, \Delta}\,(?_w)}{\vdash A \otimes B, ?C, \Gamma, \Delta}\,(\otimes)$
$\otimes - \forall - 1$	X not free in $A{\otimes}B,\Gamma,\Delta$ $\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \forall X C, \Gamma, \Delta}\,(\forall)$	$\overset{C^{\otimes}_{\forall}}{\longrightarrow}\;\underset{C^{\forall}_{\otimes}}{\longleftarrow}$	X not free in A,Γ $\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, \forall X C, \Gamma}\,(\forall) \qquad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \forall X C, \Gamma, \Delta}\,(\otimes)$
$\otimes - \forall - 2$	X not free in $A{\otimes}B,\Gamma,\Delta$ $\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, \forall X C, \Gamma, \Delta}\,(\forall)$	$\overset{C^{\otimes}_{\forall}}{\longrightarrow}\;\underset{C^{\forall}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad X \text{ not free in } B,\Delta \;\; \dfrac{\overset{\phi}{\vdash B, C, \Delta}}{\vdash B, \forall X C, \Delta}\,(\forall)}{\vdash A \otimes B, \forall X C, \Gamma, \Delta}\,(\otimes)$

Table 8: Rule commutation $\overset{r}{\longmapsto}$ (Part 4/16)

$\otimes - \exists - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, C[D/X], \Gamma, \Delta}(\otimes)}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}(\exists)$	$\overset{C^{\otimes}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma}}{\vdash A, \exists XC, \Gamma}(\exists) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}(\otimes)$
$\otimes - \exists - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, C[D/X], \Delta}}{\vdash A \otimes B, C[D/X], \Gamma, \Delta}(\otimes)}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}(\exists)$	$\overset{C^{\otimes}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, C[D/X], \Delta}}{\vdash B, \exists XC, \Delta}(\exists)}{\vdash A \otimes B, \exists XC, \Gamma, \Delta}(\otimes)$
$\otimes - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes) \quad \overset{\tau}{\vdash \Sigma}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(mix_2)$	$\overset{C^{\otimes}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash \Sigma}}{\vdash A, \Gamma, \Sigma}(mix_2) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(\otimes)$
$\otimes - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(mix_2)$	$\overset{C^{\otimes}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}(mix_2) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(\otimes)$
$\otimes - mix_2 - 3$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes) \quad \overset{\tau}{\vdash \Sigma}}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(mix_2)$	$\overset{C^{\otimes}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, \Delta} \quad \overset{\tau}{\vdash \Sigma}}{\vdash B, \Delta, \Sigma}(mix_2)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(\otimes)$
$\otimes - mix_2 - 4$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}(\otimes)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(mix_2)$	$\overset{C^{\otimes}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\phi}{\vdash A, \Delta} \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash B, \Gamma, \Sigma}(mix_2)}{\vdash A \otimes B, \Gamma, \Delta, \Sigma}(\otimes)$
$\otimes - \cup - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes) \quad \dfrac{\overset{\tau}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}(\cup)$	$\overset{C^{\otimes}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash A, \Gamma}}{\vdash A, \Gamma}(\cup) \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes)$
$\otimes - \cup - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes) \quad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes)}{\vdash A \otimes B, \Gamma, \Delta}(\cup)$	$\overset{C^{\otimes}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{\otimes}}{\longleftarrow}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\phi}{\vdash B, \Delta} \quad \overset{\tau}{\vdash B, \Delta}}{\vdash B, \Delta}(\cup)}{\vdash A \otimes B, \Gamma, \Delta}(\otimes)$
$\otimes - \emptyset - 1$	$\dfrac{}{\vdash A \otimes B, \Gamma, \Delta}(\emptyset)$	$\overset{C^{\otimes}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{\otimes}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash A, \Gamma}(\emptyset) \quad \overset{\pi}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}(\otimes)$

Table 9: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 5/16)

$\otimes - \emptyset - 2$	$\dfrac{}{\vdash A \otimes B, \Gamma, \Delta}\,(\emptyset)$	$\underset{C^{\emptyset}_{\otimes}}{\overset{C^{\otimes}_{\emptyset}}{\rightleftarrows}}$	$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{}{\vdash B, \Delta}\,(\emptyset)}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes)$
$\bot - \bot$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot_1, \Gamma}\,(\bot_1)}{\vdash \bot_1, \bot_2, \Gamma}\,(\bot_2)$	$\xrightarrow{C^{\bot}_{\bot}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot_2, \Gamma}\,(\bot_2)}{\vdash \bot_1, \bot_2, \Gamma}\,(\bot_2)$
$\bot - \&$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, \bot, \Gamma}\,(\bot) \qquad \dfrac{\overset{\phi}{\vdash B, \Gamma}}{\vdash B, \bot, \Gamma}\,(\bot)}{\vdash A \,\&\, B, \bot, \Gamma}\,(\&)$	$\underset{C^{\&}_{\bot}}{\overset{C^{\bot}_{\&}}{\rightleftarrows}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash B, \Gamma}}{\vdash A \,\&\, B, \Gamma}\,(\&)}{\vdash A \,\&\, B, \bot, \Gamma}\,(\bot)$
$\bot - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, \bot, \Gamma}\,(\bot)}{\vdash A \oplus B, \bot, \Gamma}\,(\oplus_1)$	$\underset{C^{\oplus_1}_{\bot}}{\overset{C^{\bot}_{\oplus_1}}{\rightleftarrows}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, \bot, \Gamma}\,(\bot)$
$\bot - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash B, \bot, \Gamma}\,(\bot)}{\vdash A \oplus B, \bot, \Gamma}\,(\oplus_2)$	$\underset{C^{\oplus_2}_{\bot}}{\overset{C^{\bot}_{\oplus_2}}{\rightleftarrows}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, \bot, \Gamma}\,(\bot)$
$\bot - \top$	$\dfrac{}{\vdash \top, \bot, \Gamma}\,(\top)$	$\underset{C^{\top}_{\bot}}{\overset{C^{\bot}_{\top}}{\rightleftarrows}}$	$\dfrac{\dfrac{}{\vdash \top, \Gamma}\,(\top)}{\vdash \top, \bot, \Gamma}\,(\bot)$
$\bot - ?_d$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \bot, A, \Gamma}\,(\bot)}{\vdash \bot, ?A, \Gamma}\,(?_d)$	$\underset{C^{?_d}_{\bot}}{\overset{C^{\bot}_{?_d}}{\rightleftarrows}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d)}{\vdash \bot, ?A, \Gamma}\,(\bot)$
$\bot - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash \bot, ?A, ?A, \Gamma}\,(\bot)}{\vdash \bot, ?A, \Gamma}\,(?_c)$	$\underset{C^{?_c}_{\bot}}{\overset{C^{\bot}_{?_c}}{\rightleftarrows}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c)}{\vdash \bot, ?A, \Gamma}\,(\bot)$
$\bot - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}\,(\bot)}{\vdash \bot, ?A, \Gamma}\,(?_w)$	$\underset{C^{?_w}_{\bot}}{\overset{C^{\bot}_{?_w}}{\rightleftarrows}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w)}{\vdash \bot, ?A, \Gamma}\,(\bot)$

Table 10: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 6/16)

$\bot - \forall$	X not free in $\bot,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, A, \Gamma}(\bot)}{\vdash \bot, \forall X A, \Gamma}(\forall)$	$\overset{C_{\forall}^{\bot}}{\longrightarrow}$ $\underset{C_{\bot}^{\forall}}{\longleftarrow}$	$\dfrac{X \text{ not free in } \Gamma\ \dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \forall X A, \Gamma}(\forall)}{\vdash \bot, \forall X A, \Gamma}(\bot)$
$\bot - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X], \Gamma}}{\vdash \bot, A[B/X], \Gamma}(\bot)}{\vdash \bot, \exists X A, \Gamma}(\exists)$	$\overset{C_{\exists}^{\bot}}{\longrightarrow}$ $\underset{C_{\bot}^{\exists}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X], \Gamma}}{\vdash \exists X A, \Gamma}(\exists)}{\vdash \bot, \exists X A, \Gamma}(\bot)$
$\bot - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot) \quad \overset{\phi}{\vdash \Delta}}{\vdash \bot, \Gamma, \Delta}(mix_2)$	$\overset{C_{mix_2}^{\bot}}{\longrightarrow}$ $\underset{C_{\bot}^{mix_2}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}(mix_2)}{\vdash \bot, \Gamma, \Delta}(\bot)$
$\bot - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash \bot, \Delta}(\bot)}{\vdash \bot, \Gamma, \Delta}(mix_2)$	$\overset{C_{mix_2}^{\bot}}{\longrightarrow}$ $\underset{C_{\bot}^{mix_2}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}(mix_2)}{\vdash \bot, \Gamma, \Delta}(\bot)$
$\bot - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot) \quad \dfrac{\overset{\phi}{\vdash \Gamma}}{\vdash \bot, \Gamma}(\bot)}{\vdash \bot, \Gamma}(\cup)$	$\overset{C_{\cup}^{\bot}}{\longrightarrow}$ $\underset{C_{\bot}^{\cup}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Gamma}}{\vdash \Gamma}(\cup)}{\vdash \bot, \Gamma}(\bot)$
$\bot - \emptyset$	$\dfrac{}{\vdash \bot, \Gamma}(\emptyset)$	$\overset{C_{\emptyset}^{\bot}}{\longrightarrow}$ $\underset{C_{\bot}^{\emptyset}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash \Gamma}(\emptyset)}{\vdash \bot, \Gamma}(\bot)$
$\& - \&$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \mathbin{\&} B, C, \Gamma}(\&) \quad \dfrac{\overset{\tau}{\vdash A, D, \Gamma} \quad \overset{\mu}{\vdash B, D, \Gamma}}{\vdash A \mathbin{\&} B, D, \Gamma}(\&)}{\vdash A \mathbin{\&} B, C \mathbin{\&} D, \Gamma}(\&)$	$\overset{C_{\&}^{\&}}{\longrightarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\tau}{\vdash A, D, \Gamma}}{\vdash A, C \mathbin{\&} D, \Gamma}(\&) \quad \dfrac{\overset{\phi}{\vdash B, C, \Gamma} \quad \overset{\mu}{\vdash B, D, \Gamma}}{\vdash B, C \mathbin{\&} D, \Gamma}(\&)}{\vdash A \mathbin{\&} B, C \mathbin{\&} D, \Gamma}(\&)$
$\& - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \mathbin{\&} B, C, \Gamma}(\&)}{\vdash A \mathbin{\&} B, C \oplus D, \Gamma}(\oplus_1)$	$\overset{C_{\oplus_1}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{\oplus_1}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, C \oplus D, \Gamma}(\oplus_1) \quad \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, C \oplus D, \Gamma}(\oplus_1)}{\vdash A \mathbin{\&} B, C \oplus D, \Gamma}(\&)$
$\& - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma} \quad \overset{\phi}{\vdash B, D, \Gamma}}{\vdash A \mathbin{\&} B, D, \Gamma}(\&)}{\vdash A \mathbin{\&} B, C \oplus D, \Gamma}(\oplus_2)$	$\overset{C_{\oplus_2}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{\oplus_2}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A, C \oplus D, \Gamma}(\oplus_2) \quad \dfrac{\overset{\phi}{\vdash B, D, \Gamma}}{\vdash B, C \oplus D, \Gamma}(\oplus_2)}{\vdash A \mathbin{\&} B, C \oplus D, \Gamma}(\&)$
$\& - \top$	$\dfrac{}{\vdash A \mathbin{\&} B, \top, \Gamma}(\top)$	$\overset{C_{\top}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{\top}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash A, \top, \Gamma}(\top) \quad \dfrac{}{\vdash B, \top, \Gamma}(\top)}{\vdash A \mathbin{\&} B, \top, \Gamma}(\&)$
$\& - ?_d$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \mathbin{\&} B, C, \Gamma}(\&)}{\vdash A \mathbin{\&} B, ?C, \Gamma}(?_d)$	$\overset{C_{?_d}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, ?C, \Gamma}(?_d) \quad \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, ?C, \Gamma}(?_d)}{\vdash A \mathbin{\&} B, C, \Gamma}(\&)$
$\& - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma} \quad \overset{\phi}{\vdash B, ?C, ?C, \Gamma}}{\vdash A \mathbin{\&} B, ?C, ?C, \Gamma}(\&)}{\vdash A \mathbin{\&} B, ?C, \Gamma}(?_c)$	$\overset{C_{?_c}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A, ?C, \Gamma}(?_c) \quad \dfrac{\overset{\phi}{\vdash B, ?C, ?C, \Gamma}}{\vdash B, ?C, \Gamma}(?_c)}{\vdash A \mathbin{\&} B, C, \Gamma}(\&)$
$\& - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Gamma}}{\vdash A \mathbin{\&} B, \Gamma}(\&)}{\vdash A \mathbin{\&} B, ?C, \Gamma}(?_w)$	$\overset{C_{?_w}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{?_w}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?C, \Gamma}(?_w) \quad \dfrac{\overset{\phi}{\vdash B, \Gamma}}{\vdash B, ?C, \Gamma}(?_w)}{\vdash A \mathbin{\&} B, C, \Gamma}(\&)$
$\& - \forall$	X not free in $A \& B,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma} \quad \overset{\phi}{\vdash B, C, \Gamma}}{\vdash A \mathbin{\&} B, C, \Gamma}(\&)}{\vdash A \mathbin{\&} B, \forall X C, \Gamma}(\forall)$	$\overset{C_{\forall}^{\&}}{\longrightarrow}$ $\underset{C_{\&}^{\forall}}{\longleftarrow}$	$\dfrac{X \text{ not free in } A,\Gamma\ \dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, \forall X C, \Gamma}(\forall) \quad X \text{ not free in } B,\Gamma\ \dfrac{\overset{\phi}{\vdash B, C, \Gamma}}{\vdash B, \forall X C, \Gamma}(\forall)}{\vdash A \mathbin{\&} B, \forall X C, \Gamma}(\&)$

Table 11: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 7/16)

$\& - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma} \quad \overset{\phi}{\vdash B, C[D/X], \Gamma}}{\vdash A \& B, C[D/X], \Gamma}\,(\&)}{\vdash A \& B, \exists X C, \Gamma}\,(\exists)$	$\xrightarrow{C^{\&}_{\exists}}$ $\xleftarrow[C^{\exists}_{\&}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X], \Gamma}}{\vdash A, \exists X C, \Gamma}\,(\exists) \quad \dfrac{\overset{\phi}{\vdash B, C[D/X], \Gamma}}{\vdash B, \exists X C, \Gamma}\,(\exists)}{\vdash A \& B, \exists X C, \Gamma}\,(\&)$
$\& - mix_2 - 1$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Delta}}{\vdash A \& B, \Delta}\,(\&)}{\vdash A \& B, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\&}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\&}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \quad \dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\tau}{\vdash B, \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \& B, \Gamma, \Delta}\,(\&)$
$\& - mix_2 - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash B, \Gamma}}{\vdash A \& B, \Gamma}\,(\&) \quad \overset{\phi}{\vdash \Delta}}{\vdash A \& B, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\&}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\&}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2) \quad \dfrac{\overset{\tau}{\vdash B, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \& B, \Gamma, \Delta}\,(\&)$
$\& - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash B, \Gamma}}{\vdash A \& B, \Gamma}\,(\&) \quad \dfrac{\overset{\tau}{\vdash A, \Gamma} \quad \overset{\mu}{\vdash B, \Gamma}}{\vdash A \& B, \Gamma}\,(\&)}{\vdash A \& B, \Gamma}\,(\cup)$	$\xrightarrow{C^{\&}_{\cup}}$ $\xleftarrow[C^{\cup}_{\&}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup) \quad \dfrac{\overset{\phi}{\vdash B, \Gamma} \quad \overset{\mu}{\vdash B, \Gamma}}{\vdash B, \Gamma}\,(\cup)}{\vdash A \& B, \Gamma}\,(\&)$
$\& - \emptyset$	$\dfrac{}{\vdash A \& B, \Gamma}\,(\emptyset)$	$\xrightarrow{C^{\&}_{\emptyset}}$ $\xleftarrow[C^{\emptyset}_{\&}]{}$	$\dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset) \quad \dfrac{}{\vdash B, \Gamma}\,(\emptyset)}{\vdash A \& B, \Gamma}\,(\&)$
$\oplus_1 - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A \oplus B, C, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_1)$	$\xrightarrow{C^{\oplus_1}_{\oplus_1}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_1)$
$\oplus_1 - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A \oplus B, D, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_2)$	$\xrightarrow{C^{\oplus_1}_{\oplus_2}}$ $\xleftarrow[C^{\oplus_2}_{\oplus_1}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, D, \Gamma}}{\vdash A, C \oplus D, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, C \oplus D, \Gamma}\,(\oplus_1)$
$\oplus_1 - \top$	$\dfrac{}{\vdash A \oplus B, \top, \Gamma}\,(\top)$	$\xrightarrow{C^{\oplus_1}_{\top}}$ $\xleftarrow[C^{\top}_{\oplus_1}]{}$	$\dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top)}{\vdash A \oplus B, \top, \Gamma}\,(\oplus_1)$
$\oplus_1 - ?_d$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A \oplus B, C, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(?_d)$	$\xrightarrow{C^{\oplus_1}_{?_d}}$ $\xleftarrow[C^{?_d}_{\oplus_1}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_d)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_1)$
$\oplus_1 - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A \oplus B, ?C, ?C, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(?_c)$	$\xrightarrow{C^{\oplus_1}_{?_c}}$ $\xleftarrow[C^{?_c}_{\oplus_1}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?C, ?C, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_c)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_1)$
$\oplus_1 - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_1)}{\vdash A \oplus B, ?C, \Gamma}\,(?_w)$	$\xrightarrow{C^{\oplus_1}_{?_w}}$ $\xleftarrow[C^{?_w}_{\oplus_1}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?C, \Gamma}\,(?_w)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_1)$

Table 12: Rule commutation $\overset{r}{\mapsto}$ (Part 8/16)

$\oplus_1 - \forall$	X not free in $A \oplus B,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A\oplus B, C, \Gamma}\,(\oplus_1)}{\vdash A\oplus B,\forall XC,\Gamma}\,(\forall)$	$\overset{C^{\oplus_1}_{\forall}}{\longrightarrow}$ $\underset{C^{\forall}_{\oplus_1}}{\longleftarrow}$	X not free in A,Γ $\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A,\forall XC,\Gamma}\,(\forall)}{\vdash A\oplus B,\forall XC,\Gamma}\,(\oplus_1)$
$\oplus_1 - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X],\Gamma}}{\vdash A\oplus B, C[D/X], \Gamma}\,(\oplus_1)}{\vdash A\oplus B,\exists XC,\Gamma}\,(\exists)$	$\overset{C^{\oplus_1}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{\oplus_1}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, C[D/X],\Gamma}}{\vdash A,\exists XC,\Gamma}\,(\exists)}{\vdash A\oplus B,\exists XC,\Gamma}\,(\oplus_1)$
$\oplus_1 - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_1) \quad \overset{\phi}{\vdash \Delta}}{\vdash A\oplus B,\Gamma,\Delta}\,(mix_2)$	$\overset{C^{\oplus_1}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\oplus_1}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A,\Gamma,\Delta}\,(mix_2)}{\vdash A\oplus B,\Gamma,\Delta}\,(\oplus_1)$
$\oplus_1 - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A,\Delta}}{\vdash A\oplus B,\Delta}\,(\oplus_1)}{\vdash A\oplus B,\Gamma,\Delta}\,(mix_2)$	$\overset{C^{\oplus_1}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\oplus_1}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A,\Delta}}{\vdash A,\Gamma,\Delta}\,(mix_2)}{\vdash A\oplus B,\Gamma,\Delta}\,(\oplus_1)$
$\oplus_1 - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_1) \quad \dfrac{\overset{\phi}{\vdash A,\Gamma}}{\vdash A\oplus B,\Gamma}\,(\oplus_1)}{\vdash A\oplus B,\Gamma}\,(\cup)$	$\overset{C^{\oplus_1}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{\oplus_1}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma} \quad \overset{\phi}{\vdash A,\Gamma}}{\vdash A,\Gamma}\,(\cup)}{\vdash A\oplus B,\Gamma}\,(\oplus_1)$
$\oplus_1 - \emptyset$	$\dfrac{}{\vdash A\oplus B,\Gamma}\,(\emptyset)$	$\overset{C^{\oplus_1}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{\oplus_1}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash A,\Gamma}\,(\emptyset)}{\vdash A\oplus B,\Gamma}\,(\oplus_1)$
$\oplus_2 - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B,D,\Gamma}}{\vdash A\oplus B,D,\Gamma}\,(\oplus_2)}{\vdash A\oplus B,C\oplus D,\Gamma}\,(\oplus_2)$	$\overset{C^{\oplus_2}_{\oplus_2}}{\longrightarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B,D,\Gamma}}{\vdash B,C\oplus D,\Gamma}\,(\oplus_2)}{\vdash A\oplus B,C\oplus D,\Gamma}\,(\oplus_2)$
$\oplus_2 - \top$	$\dfrac{}{\vdash A\oplus B,\top,\Gamma}\,(\top)$	$\overset{C^{\oplus_2}_{\top}}{\longrightarrow}$ $\underset{C^{\top}_{\oplus_2}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash B,\top,\Gamma}\,(\top)}{\vdash A\oplus B,\top,\Gamma}\,(\oplus_2)$
$\oplus_2 - ?_d$	$\dfrac{\dfrac{\overset{\pi}{\vdash B,C,\Gamma}}{\vdash A\oplus B,C,\Gamma}\,(\oplus_2)}{\vdash A\oplus B,?C,\Gamma}\,(?_d)$	$\overset{C^{\oplus_2}_{?_d}}{\longrightarrow}$ $\underset{C^{?_d}_{\oplus_2}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B,C,\Gamma}}{\vdash B,?C,\Gamma}\,(?_d)}{\vdash A\oplus B,?C,\Gamma}\,(\oplus_2)$

Table 13: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 9/16)

$\oplus_2 - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, ?C, ?C, \Gamma}}{\vdash A \oplus B, ?C, ?C, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, ?C, \Gamma}\,(?_c)$	$\xrightarrow{C^{\oplus_2}_{?_c}}$ $\xleftarrow[C^{?_c}_{\oplus_2}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, ?C, ?C, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_c)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_2)$
$\oplus_2 - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, ?C, \Gamma}\,(?_w)$	$\xrightarrow{C^{\oplus_2}_{?_w}}$ $\xleftarrow[C^{?_w}_{\oplus_2}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash B, ?C, \Gamma}\,(?_w)}{\vdash A \oplus B, ?C, \Gamma}\,(\oplus_2)$
$\oplus_2 - \forall$	X not free in $A \oplus B, \Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash B, C, \Gamma}}{\vdash A \oplus B, C, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, \forall XC, \Gamma}\,(\forall)$	$\xrightarrow{C^{\oplus_2}_{\forall}}$ $\xleftarrow[C^{\forall}_{\oplus_2}]{}$	X not free in B, Γ $\dfrac{\dfrac{\overset{\pi}{\vdash B, C, \Gamma}}{\vdash B, \forall XC, \Gamma}\,(\forall)}{\vdash A \oplus B, \forall XC, \Gamma}\,(\oplus_2)$
$\oplus_2 - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, C[D/X], \Gamma}}{\vdash A \oplus B, C[D/X], \Gamma}\,(\oplus_2)}{\vdash A \oplus B, \exists XC, \Gamma}\,(\exists)$	$\xrightarrow{C^{\oplus_2}_{\exists}}$ $\xleftarrow[C^{\exists}_{\oplus_2}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, C[D/X], \Gamma}}{\vdash B, \exists XC, \Gamma}\,(\exists)}{\vdash A \oplus B, \exists XC, \Gamma}\,(\oplus_2)$
$\oplus_2 - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2) \quad \overset{\phi}{\vdash \Delta}}{\vdash A \oplus B, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\oplus_2}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\oplus_2}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(\oplus_2)$
$\oplus_2 - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash A \oplus B, \Delta}\,(\oplus_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\oplus_2}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\oplus_2}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash B, \Delta}}{\vdash B, \Gamma, \Delta}\,(mix_2)}{\vdash A \oplus B, \Gamma, \Delta}\,(\oplus_2)$
$\oplus_2 - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2) \quad \dfrac{\overset{\phi}{\vdash B, \Gamma}}{\vdash A \oplus B, \Gamma}\,(\oplus_2)}{\vdash A \oplus B, \Gamma}\,(\cup)$	$\xrightarrow{C^{\oplus_2}_{\cup}}$ $\xleftarrow[C^{\cup}_{\oplus_2}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \overset{\phi}{\vdash B, \Gamma}}{\vdash B, \Gamma}\,(\cup)}{\vdash A \oplus B, \Gamma}\,(\oplus_2)$
$\oplus_2 - \emptyset$	$\dfrac{}{\vdash A \oplus B, \Gamma}\,(\emptyset)$	$\xrightarrow{C^{\oplus_2}_{\emptyset}}$ $\xleftarrow[C^{\emptyset}_{\oplus_2}]{}$	$\dfrac{\dfrac{}{\vdash B, \Gamma}\,(\emptyset)}{\vdash A \oplus B, \Gamma}\,(\oplus_2)$
$\top - \top$	$\dfrac{}{\vdash \top_1, \top_2, \Gamma}\,(\top_2)$	$\xrightarrow{C^{\top}_{\top}}$	$\dfrac{}{\vdash \top_1, \top_2, \Gamma}\,(\top_1)$
$\top - ?_d$	$\dfrac{\dfrac{}{\vdash \top, A, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_d)$	$\xrightarrow{C^{\top}_{?_d}}$ $\xleftarrow[C^{?_d}_{\top}]{}$	$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top)$

Table 14: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 10/16)

$\top - ?_c$	$\dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_c)$	$\overset{C^{\top}_{?_c}}{\longrightarrow}$ $\underset{C^{?_c}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top)$
$\top - ?_w$	$\dfrac{\dfrac{}{\vdash \top, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_w)$	$\overset{C^{\top}_{?_w}}{\longrightarrow}$ $\underset{C^{?_w}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top)$
$\top - \forall$	X not free in $\top,\Gamma$ $\dfrac{\dfrac{}{\vdash \top, A, \Gamma}\,(\top)}{\vdash \top, \forall X A, \Gamma}\,(\forall)$	$\overset{C^{\top}_{\forall}}{\longrightarrow}$ $\underset{C^{\forall}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, \forall X A, \Gamma}\,(\top)$
$\top - \exists$	$\dfrac{\dfrac{}{\vdash \top, A[B/X], \Gamma}\,(\top)}{\vdash \top, \exists X A, \Gamma}\,(\exists)$	$\overset{C^{\top}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, \exists X A, \Gamma}\,(\top)$
$\top - mix_2 - 1$	$\dfrac{\dfrac{}{\vdash \top, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash \Delta}}{\vdash \top, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{\top}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, \Gamma, \Delta}\,(\top)$
$\top - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash \top, \Delta}\,(\top)}{\vdash \top, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{\top}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, \Gamma, \Delta}\,(\top)$
$\top - \cup$	$\dfrac{\dfrac{}{\vdash \top, \Gamma}\,(\top) \qquad \dfrac{}{\vdash \top, \Gamma}\,(\top)}{\vdash \top, \Gamma}\,(\cup)$	$\overset{C^{\top}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, \Gamma}\,(\top)$
$\top - \emptyset$	$\dfrac{}{\vdash \top, \Gamma}\,(\emptyset)$	$\overset{C^{\top}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{\top}}{\longleftarrow}$	$\dfrac{}{\vdash \top, \Gamma}\,(\top)$
$?_d - ?_d$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash ?A, B, \Gamma}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(?_d)$	$\overset{C^{?_d}_{?_d}}{\longrightarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(?_d)$
$?_d - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, \Gamma}}{\vdash ?A, ?B, ?B, \Gamma}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(?_c)$	$\overset{C^{?_d}_{?_c}}{\longrightarrow}$ $\underset{C^{?_c}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_c)}{\vdash ?A, ?B, \Gamma}\,(?_d)$
$?_d - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d)}{\vdash ?A, ?B, \Gamma}\,(?_w)$	$\overset{C^{?_d}_{?_w}}{\longrightarrow}$ $\underset{C^{?_w}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, ?B, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_d)$

Table 15: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 11/16)

$?_d - \forall$	X not free in $?A,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash ?A, B, \Gamma}\,(?_d)}{\vdash ?A, \forall X B, \Gamma}\,(\forall)$	$\overset{C^{?_d}_{\forall}}{\longrightarrow}$ $\underset{C^{\forall}_{?_d}}{\longleftarrow}$	X not free in A,Γ $\dfrac{\dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, \forall X B, \Gamma}\,(\forall)}{\vdash ?A, \forall X B, \Gamma}\,(?_d)$
$?_d - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B[C/X], \Gamma}}{\vdash A, B[C/X], \Gamma}\,(?_d)}{\vdash ?A, \exists X B, \Gamma}\,(\exists)$	$\overset{C^{?_d}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, B[C/X], \Gamma}}{\vdash A, \exists X B, \Gamma}\,(\exists)}{\vdash ?A, \exists X B, \Gamma}\,(?_d)$
$?_d - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d) \quad \overset{\phi}{\vdash \Delta}}{\vdash ?A, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{?_d}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_d)$
$?_d - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash A, \Delta}}{\vdash ?A, \Delta}\,(?_d)}{\vdash ?A, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{?_d}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_d)$
$?_d - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d) \quad \dfrac{\overset{\phi}{\vdash A, \Gamma}}{\vdash ?A, \Gamma}\,(?_d)}{\vdash ?A, \Gamma}\,(\cup)$	$\overset{C^{?_d}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup)}{\vdash ?A, \Gamma}\,(?_d)$
$?_d - \emptyset$	$\dfrac{}{\vdash ?A, \Gamma}\,(\emptyset)$	$\overset{C^{?_d}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{?_d}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset)}{\vdash ?A, \Gamma}\,(?_d)$
$?_c - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, ?B, ?B, \Gamma}}{\vdash ?A, ?B, ?B, \Gamma}\,(?_c)}{\vdash ?A, ?B, \Gamma}\,(?_c)$	$\overset{C^{?_c}_{?_c}}{\longrightarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, ?B, ?B, \Gamma}}{\vdash ?A, ?A, ?B, \Gamma}\,(?_c)}{\vdash ?A, ?B, \Gamma}\,(?_c)$
$?_c - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c)}{\vdash ?A, ?B, \Gamma}\,(?_w)$	$\overset{C^{?_c}_{?_w}}{\longrightarrow}$ $\underset{C^{?_w}_{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, ?A, ?B, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_c)$
$?_c - \forall$	X not free in $?A,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B, \Gamma}}{\vdash ?A, B, \Gamma}\,(?_c)}{\vdash ?A, \forall X B, \Gamma}\,(\forall)$	$\overset{C^{?_c}_{\forall}}{\longrightarrow}$ $\underset{C^{\forall}_{?_c}}{\longleftarrow}$	X not free in $?A,?A,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B, \Gamma}}{\vdash ?A, ?A, \forall X B, \Gamma}\,(\forall)}{\vdash ?A, \forall X B, \Gamma}\,(?_c)$

Table 16: Rule commutation $\overset{r}{\longmapsto}$ (Part 12/16)

$?_c - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, ?A, B[C/X], \Gamma}\,(?_c)}{\vdash ?A, \exists XB, \Gamma}\,(\exists)$	$\overset{C^{?_c}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, B[C/X], \Gamma}}{\vdash ?A, ?A, \exists XB, \Gamma}\,(\exists)}{\vdash ?A, \exists XB, \Gamma}\,(?_c)$
$?_c - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c) \quad \overset{\phi}{\vdash \Delta}}{\vdash ?A, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{?_c}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash ?A, ?A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_c)$
$?_c - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash ?A, ?A, \Delta}}{\vdash ?A, \Delta}\,(?_c)}{\vdash ?A, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{?_c}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash ?A, ?A, \Delta}}{\vdash ?A, ?A, \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_c)$
$?_c - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c) \quad \dfrac{\overset{\phi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, \Gamma}\,(?_c)}{\vdash ?A, \Gamma}\,(\cup)$	$\overset{C^{?_c}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, ?A, \Gamma} \quad \overset{\phi}{\vdash ?A, ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(\cup)}{\vdash ?A, \Gamma}\,(?_c)$
$?_c - \emptyset$	$\dfrac{}{\vdash ?A, \Gamma}\,(\emptyset)$	$\overset{C^{?_c}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{?_c}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash ?A, ?A, \Gamma}\,(\emptyset)}{\vdash ?A, \Gamma}\,(?_c)$
$?_w - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_w)$	$\overset{C^{?_w}_{?_w}}{\longrightarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?B, \Gamma}\,(?_w)}{\vdash ?A, ?B, \Gamma}\,(?_w)$
$?_w - \forall$	X not free in $?A,\Gamma$ $\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash ?A, B, \Gamma}\,(?_w)}{\vdash ?A, \forall XB, \Gamma}\,(\forall)$	$\overset{C^{?_w}_{\forall}}{\longrightarrow}$ $\underset{C^{\forall}_{?_w}}{\longleftarrow}$	X not free in Γ $\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma}}{\vdash \forall XB, \Gamma}\,(\forall)}{\vdash ?A, \forall XB, \Gamma}\,(?_w)$
$?_w - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash B[C/X], \Gamma}}{\vdash B[C/X], \Gamma}\,(?_w)}{\vdash ?A, \exists XB, \Gamma}\,(\exists)$	$\overset{C^{?_w}_{\exists}}{\longrightarrow}$ $\underset{C^{\exists}_{?_w}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash B[C/X], \Gamma}}{\vdash \exists XB, \Gamma}\,(\exists)}{\vdash ?A, \exists XB, \Gamma}\,(?_w)$
$?_w - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w) \quad \overset{\phi}{\vdash \Delta}}{\vdash ?A, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{?_w}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{?_w}}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_w)$

Table 17: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 13/16)

$?_w - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \dfrac{\overset{\phi}{\vdash \Delta}}{\vdash ?A, \Delta}\,(?_w)}{\vdash ?A, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{?_w}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{?_w}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash ?A, \Gamma, \Delta}\,(?_w)$
$?_w - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w) \quad \dfrac{\overset{\phi}{\vdash \Gamma}}{\vdash ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(\cup)$	$\xrightarrow{C^{?_w}_{\cup}}$ $\xleftarrow[C^{\cup}_{?_w}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup)}{\vdash ?A, \Gamma}\,(?_w)$
$?_w - \emptyset$	$\dfrac{}{\vdash ?A, \Gamma}\,(\emptyset)$	$\xrightarrow{C^{?_w}_{\emptyset}}$ $\xleftarrow[C^{\emptyset}_{?_w}]{}$	$\dfrac{\dfrac{}{\vdash \Gamma}\,(\emptyset)}{\vdash ?A, \Gamma}\,(?_w)$
$\forall - \forall$	$\text{$Y$ not free in $\forall XA,\Gamma$}\ \dfrac{\text{X not free in B,Γ}\ \dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash \forall XA, B, \Gamma}\,(\forall)}{\vdash \forall XA, \forall YB, \Gamma}\,(\forall)$	$\xrightarrow{C^{\forall}_{\forall}}$	$\text{$X$ not free in $\forall YB,\Gamma$}\ \dfrac{\text{Y not free in A,Γ}\ \dfrac{\overset{\pi}{\vdash A, B, \Gamma}}{\vdash A, \forall YB, \Gamma}\,(\forall)}{\vdash \forall XA, \forall YB, \Gamma}\,(\forall)$
$\forall - \exists$	$\dfrac{\text{$X$ not free in $B[C/Y],\Gamma$}\ \dfrac{\overset{\pi}{\vdash A, B[C/Y], \Gamma}}{\vdash \forall XA, B[C/Y], \Gamma}\,(\forall)}{\vdash \forall XA, \exists YB, \Gamma}\,(\exists)$	$\xrightarrow{C^{\forall}_{\exists}}$ $\xleftarrow[C^{\exists}_{\forall}]{}$	$\text{$X$ not free in $\exists YB,\Gamma$}\ \dfrac{\dfrac{\overset{\pi}{\vdash A, B[C/Y], \Gamma}}{\vdash A, \exists YB, \Gamma}\,(\exists)}{\vdash \forall XA, \exists YB, \Gamma}\,(\forall)$
$\forall - mix_2 - 1$	$\dfrac{\text{$X$ not free in Γ}\ \dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \forall XA, \Gamma}\,(\forall) \quad \overset{\phi}{\vdash \Delta}}{\vdash \forall XA, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\forall}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\forall}]{}$	$\text{$X$ not free in Γ,Δ}\ \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash \forall XA, \Gamma, \Delta}\,(\forall)$
$\forall - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \quad \text{X not free in Δ}\ \dfrac{\overset{\phi}{\vdash A, \Delta}}{\vdash \forall XA, \Delta}\,(\forall)}{\vdash \forall XA, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\forall}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\forall}]{}$	$\text{$X$ not free in Γ,Δ}\ \dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash A, \Gamma, \Delta}\,(mix_2)}{\vdash \forall XA, \Gamma, \Delta}\,(\forall)$
$\forall - \cup$	$\dfrac{\text{$X$ not free in Γ}\ \dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash \forall XA, \Gamma}\,(\forall) \quad \text{X not free in Γ}\ \dfrac{\overset{\phi}{\vdash A, \Gamma}}{\vdash \forall XA, \Gamma}\,(\forall)}{\vdash \forall XA, \Gamma}\,(\cup)$	$\xrightarrow{C^{\forall}_{\cup}}$ $\xleftarrow[C^{\cup}_{\forall}]{}$	$\text{$X$ not free in Γ}\ \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\phi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(\cup)}{\vdash \forall XA, \Gamma}\,(\forall)$
$\forall - \emptyset$	$\dfrac{}{\vdash \forall XA, \Gamma}\,(\emptyset)$	$\xrightarrow{C^{\forall}_{\emptyset}}$ $\xleftarrow[C^{\emptyset}_{\forall}]{}$	$\text{$X$ not free in Γ}\ \dfrac{\dfrac{}{\vdash A, \Gamma}\,(\emptyset)}{\vdash \forall XA, \Gamma}\,(\forall)$
$\exists - \exists$	$\dfrac{\dfrac{\overset{\pi}{\vdash A[C/X], B[D/Y], \Gamma}}{\vdash \exists XA, B[D/Y], \Gamma}\,(\exists)}{\vdash \exists XA, \exists YB, \Gamma}\,(\exists)$	$\xrightarrow{C^{\exists}_{\exists}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A[C/X], B[D/Y], \Gamma}}{\vdash A[C/X], \exists YB, \Gamma}\,(\exists)}{\vdash \exists XA, \exists YB, \Gamma}\,(\exists)$
$\exists - mix_2 - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X], \Gamma}}{\vdash \exists XA, \Gamma}\,(\exists) \quad \overset{\phi}{\vdash \Delta}}{\vdash \exists XA, \Gamma, \Delta}\,(mix_2)$	$\xrightarrow{C^{\exists}_{mix_2}}$ $\xleftarrow[C^{mix_2}_{\exists}]{}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A[B/X], \Gamma} \quad \overset{\phi}{\vdash \Delta}}{\vdash A[B/X], \Gamma, \Delta}\,(mix_2)}{\vdash \exists XA, \Gamma, \Delta}\,(\exists)$

Table 18: Rule commutation $\overset{r}{\longmapsto}$ (Part 14/16)

$\exists - mix_2 - 2$	$\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \dfrac{\stackrel{\phi}{\vdash A[B/X], \Delta}}{\vdash \exists XA, \Delta}\,(\exists)}{\vdash \exists XA, \Gamma, \Delta}\,(mix_2)$	$\overset{C^{\exists}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{\exists}}{\longleftarrow}$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash A[B/X], \Delta}}{\vdash A[B/X], \Gamma, \Delta}\,(mix_2)}{\vdash \exists XA, \Gamma, \Delta}\,(\exists)$
$\exists - \cup$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash A[B/X], \Gamma}}{\vdash \exists XA, \Gamma}\,(\exists) \quad \dfrac{\stackrel{\phi}{\vdash A[B/X], \Gamma}}{\vdash \exists XA, \Gamma}\,(\exists)}{\vdash \exists XA, \Gamma}\,(\cup)$	$\overset{C^{\exists}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{\exists}}{\longleftarrow}$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash A[B/X], \Gamma} \quad \stackrel{\phi}{\vdash A[B/X], \Gamma}}{\vdash A[B/X], \Gamma}\,(\cup)}{\vdash \exists XA, \Gamma}\,(\exists)$
$\exists - \emptyset$	$\dfrac{}{\vdash \exists XA, \Gamma}\,(\emptyset)$	$\overset{C^{\exists}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{\exists}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash A[B/X], \Gamma}\,(\emptyset)}{\vdash \exists XA, \Gamma}\,(\exists)$
$mix_2 - mix_2 - 1$	$\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \dfrac{\stackrel{\phi}{\vdash \Delta} \quad \stackrel{\tau}{\vdash \Sigma}}{\vdash \Delta, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$	$\overset{C^{mix_2}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{mix_2}}{\longleftarrow}$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \stackrel{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$
$mix_2 - mix_2 - 2$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \stackrel{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$	$\overset{C^{mix_2}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{mix_2}}{\longleftarrow}$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Sigma}\,(mix_2) \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$
$mix_2 - mix_2 - 3$	$\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \dfrac{\stackrel{\phi}{\vdash \Delta} \quad \stackrel{\tau}{\vdash \Sigma}}{\vdash \Delta, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$	$\overset{C^{mix_2}_{mix_2}}{\longrightarrow}$ $\underset{C^{mix_2}_{mix_2}}{\longleftarrow}$	$\dfrac{\stackrel{\phi}{\vdash \Delta} \quad \dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\tau}{\vdash \Sigma}}{\vdash \Gamma, \Sigma}\,(mix_2)}{\vdash \Gamma, \Delta, \Sigma}\,(mix_2)$
$mix_2 - \cup - 1$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\tau}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash \Gamma, \Delta}\,(\cup)$	$\overset{C^{mix_2}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{mix_2}}{\longleftarrow}$	$\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \dfrac{\stackrel{\phi}{\vdash \Delta} \quad \stackrel{\tau}{\vdash \Delta}}{\vdash \Delta}\,(\cup)}{\vdash \Gamma, \Delta}\,(mix_2)$
$mix_2 - \cup - 2$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2) \quad \dfrac{\stackrel{\tau}{\vdash \Gamma} \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)}{\vdash \Gamma, \Delta}\,(\cup)$	$\overset{C^{mix_2}_{\cup}}{\longrightarrow}$ $\underset{C^{\cup}_{mix_2}}{\longleftarrow}$	$\dfrac{\dfrac{\stackrel{\pi}{\vdash \Gamma} \quad \stackrel{\tau}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup) \quad \stackrel{\phi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)$
$mix_2 - \emptyset - 1$	$\dfrac{}{\vdash \Gamma, \Delta}\,(\emptyset)$	$\overset{C^{mix_2}_{\emptyset}}{\longrightarrow}$ $\underset{C^{\emptyset}_{mix_2}}{\longleftarrow}$	$\dfrac{\dfrac{}{\vdash \Gamma}\,(\emptyset) \quad \stackrel{\pi}{\vdash \Delta}}{\vdash \Gamma, \Delta}\,(mix_2)$

Table 19: Rule commutation $\overset{r}{\vdash\!\dashv}$ (Part 15/16)

$mix_2 - \emptyset - 2$	$\dfrac{}{\vdash \Gamma, \Delta}\,(\emptyset)$	$\xrightarrow{C_{\emptyset}^{mix_2}}$ $\xleftarrow[C^{\emptyset}_{mix_2}]{}$	$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash \Delta}\,(\emptyset)}{\vdash \Gamma, \Delta}\,(mix_2)$
$\cup - \cup$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\phi}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup) \qquad \dfrac{\overset{\tau}{\vdash \Gamma} \qquad \overset{\mu}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup)}{\vdash \Gamma}\,(\cup)$	$\xrightarrow{C_{\cup}^{\cup}}$	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \overset{\tau}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup) \qquad \dfrac{\overset{\phi}{\vdash \Gamma} \qquad \overset{\mu}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup)}{\vdash \Gamma}\,(\cup)$
$\cup - \emptyset$	$\dfrac{}{\vdash \Gamma}\,(\emptyset)$	$\xrightarrow{C_{\emptyset}^{\cup}}$ $\xleftarrow[C^{\emptyset}_{\cup}]{}$	$\dfrac{\dfrac{}{\vdash \Gamma}\,(\emptyset) \qquad \dfrac{}{\vdash \Gamma}\,(\emptyset)}{\vdash \Gamma}\,(\cup)$

Table 20: Rule commutation $\overset{r}{\longmapsto}$ (Part 16/16)

$! - ?_c$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B_1, ?B_2, ?\Gamma}}{\vdash !A, ?B_1, ?B_2, ?\Gamma}\,(!)}{\vdash !A, ?B_1, ?\Gamma}\,(?_c)$	$\overset{C^!_{?_c}}{\longrightarrow}$ $\underset{C^{?_c}_!}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B_1, ?B_2, ?\Gamma}}{\vdash A, ?B_1, ?\Gamma}\,(?_c)}{\vdash !A, ?B_1, ?\Gamma}\,(!)$
$! - ?_w$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w)$	$\overset{C^!_{?_w}}{\longrightarrow}$ $\underset{C^{?_w}_!}{\longleftarrow}$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$
$?_c$ associative	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1, ?A_2, ?A_3, \Gamma}}{\vdash ?A_1, ?A_2, \Gamma}\,(?_c)}{\vdash ?A_1, \Gamma}\,(?_c)$	$\divideontimes$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1, ?A_2, ?A_3, \Gamma}}{\vdash ?A_1, ?A_3, \Gamma}\,(?_c)}{\vdash ?A_1, \Gamma}\,(?_c)$
$?_c$ commutative	$\dfrac{\overset{\pi}{\vdash ?A_1, ?A_2, \Gamma}}{\vdash ?A_1, \Gamma}\,(?_c)$	$\divideontimes$	$\dfrac{\overset{\pi}{\vdash ?A_1, ?A_2, \Gamma}}{\vdash ?A_2, \Gamma}\,(?_c)$
$?_w - ?_c - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A_1, \Gamma}}{\vdash ?A_1, ?A_2, \Gamma}\,(?_w)}{\vdash ?A_1, \Gamma}\,(?_c)$	$\xrightarrow{Re}$	$\overset{\pi}{\vdash ?A_1, \Gamma}$
$?_w - ?_c - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A_2, \Gamma}}{\vdash ?A_1, ?A_2, \Gamma}\,(?_w)}{\vdash ?A_1, \Gamma}\,(?_c)$	$\xrightarrow{Re}$	$\overset{\pi}{\vdash ?A, \Gamma}$
$mix_0 - mix_2 - 1$	$\dfrac{\dfrac{}{\vdash}\,(mix_0) \qquad \overset{\pi}{\vdash \Gamma}}{\vdash \Gamma}\,(mix_2)$	$\xrightarrow{Rm}$	$\overset{\pi}{\vdash \Gamma}$
$mix_0 - mix_2 - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash}\,(mix_0)}{\vdash \Gamma}\,(mix_2)$	$\xrightarrow{Rm}$	$\overset{\pi}{\vdash \Gamma}$
$\emptyset - \cup - 1$	$\dfrac{\dfrac{}{\vdash \Gamma}\,(\emptyset) \qquad \overset{\pi}{\vdash \Gamma}}{\vdash \Gamma}\,(\cup)$	$\xrightarrow{Ra}$	$\overset{\pi}{\vdash \Gamma}$
$\emptyset - \cup - 2$	$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash \Gamma}\,(\emptyset)}{\vdash \Gamma}\,(\cup)$	$\xrightarrow{Ra}$	$\overset{\pi}{\vdash \Gamma}$

Table 21: Additional transformations

$\parr - \otimes$	$\dfrac{}{\vdash B^\perp \parr A^\perp, A \otimes B}\,(ax)$	$\xrightarrow{\eta}$	$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash B^\perp, A^\perp, A \otimes B}\,(\otimes)}{\vdash B^\perp \parr A^\perp, A \otimes B}\,(\parr)$
$\perp - 1$	$\dfrac{}{\vdash \perp, 1}\,(ax)$	$\xrightarrow{\eta}$	$\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash \perp, 1}\,(\perp)$
$\& - \oplus$	$\dfrac{}{\vdash B^\perp \,\&\, A^\perp, A \oplus B}\,(ax)$	$\xrightarrow{\eta}$	$\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash B^\perp, A \oplus B}\,(\oplus_2) \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax)}{\vdash A^\perp, A \oplus B}\,(\oplus_1)}{\vdash B^\perp \,\&\, A^\perp, A \oplus B}\,(\&)$
$\top - 0$	$\dfrac{}{\vdash \top, 0}\,(ax)$	$\xrightarrow{\eta}$	$\dfrac{}{\vdash \top, 0}\,(\top)$
$? - !$	$\dfrac{}{\vdash ?A^\perp, !A}\,(ax)$	$\xrightarrow{\eta}$	$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash ?A^\perp, A}\,(?_d)}{\vdash ?A^\perp, !A}\,(!)$
$\forall - \exists$	$\dfrac{}{\vdash \forall X A^\perp, \exists X A}\,(ax)$	$\xrightarrow{\eta}$	$X \text{ not free in } \exists X A \ \dfrac{X^+ \ \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash A^\perp, \exists X A}\,(\exists)}{\vdash \forall X A^\perp, \exists X A}\,(\forall)$

Table 22: Axiom-expansion $\xrightarrow{\eta}$