Confluence Modulo and Undecidability of Cut-Elimination*

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Abstract

Proofs in a logic are usually considered (at least) up to cut-elimination. This is the case in linear logic, that also has another equivalence relation on proofs: rule commutation. We prove that these two notions coincide in second-order linear logic and many of its fragments. Moreover, we show that equality up to rule commutation is undecidable in propositional linear logic—and so is equality up to cut-elimination.

1 Introduction

A standard result in linear logic is the admissibility of the *cut*-rule, through a rewriting procedure called *cut-elimination* that turns a derivation (*a.k.a.* proof tree) into another derivation with no *cut*-rule. Through the Curry-Howard correspondence, cut-elimination is linked to β-reduction of λ-calculus, and it is sensible to consider derivations up to cut-elimination, exactly as λ-terms are considered up to β-reduction. This is often done in semantics, especially in denotational models like coherent spaces [Gir87] or Seely categories [See89]. To consider objects up to a rewriting procedure, the royal road is having *strong normalization* and *confluence*. They yield together the *unique normal form* property, giving a *canonical* representative for each equivalence class. Alas, cut-elimination is not that well-behaved: it enjoys neither strong normalization nor confluence.

No strong normalization. The $cut - cut$ step of *cut*-elimination can be repeated *ad nauseam*:

$$\dfrac{\dfrac{\vdash A^\perp, B^\perp, \Gamma \qquad \vdash A, \Delta}{\vdash B^\perp, \Gamma, \Delta}\,(cut) \qquad \vdash B, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \longrightarrow \dfrac{\dfrac{\vdash A^\perp, B^\perp, \Gamma \qquad \vdash B, \Sigma}{\vdash A^\perp, \Gamma, \Sigma}\,(cut) \qquad \vdash A, \Delta}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

No confluence. A derivation can be rewritten to different cut-free (*i.e.* normal) forms:

$$\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \quad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes) \overset{+}{\longleftarrow} \dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \quad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \quad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(cut) \overset{+}{\longrightarrow} \dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \quad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes)$$

Consequently, the literature on cut-elimination in sequent calculus is not as developed as one may wish: there is not much more than weak normalization, *i.e.* admissibility of the *cut*-rule—see for instance [Gir95; Oka99]. Nonetheless, such a result has been proved for many variations of linear logic, see *e.g.* [EP16; Acc22; AMM25; BS25; BS26]. Naturally, not everything about cut-elimination revolves around normalization and confluence, and there are many works where cut-elimination is central, *e.g.* around session types [DeY+12] or interpolation [Sau25]. Let us mention cut-elimination has been well-studied for *proof-nets* [Gir96], the graphical syntax of linear logic. This framework is closely linked to sequent calculus, but there strong normalization and confluence (usually) hold, leading to a very rich literature on the subject for many fragments and extensions of linear logic, see *e.g.* [Tor03; LM08; DG99; Acc13; Gue+24; Pag09; Tra09; PT17; PT10].

In sequent calculus, since strong normalization and confluence fail we have to consider weaker properties. The failure of strong normalization is only due to the $cut - cut$ step: provided this step does not occur infinitely many times, one obtains strong normalization. Besides, cut-elimination is confluent *up to rule commutation*: normal forms differ only by the order in which their rules are applied—see our example above.

*Related to an extended draft [Di26].

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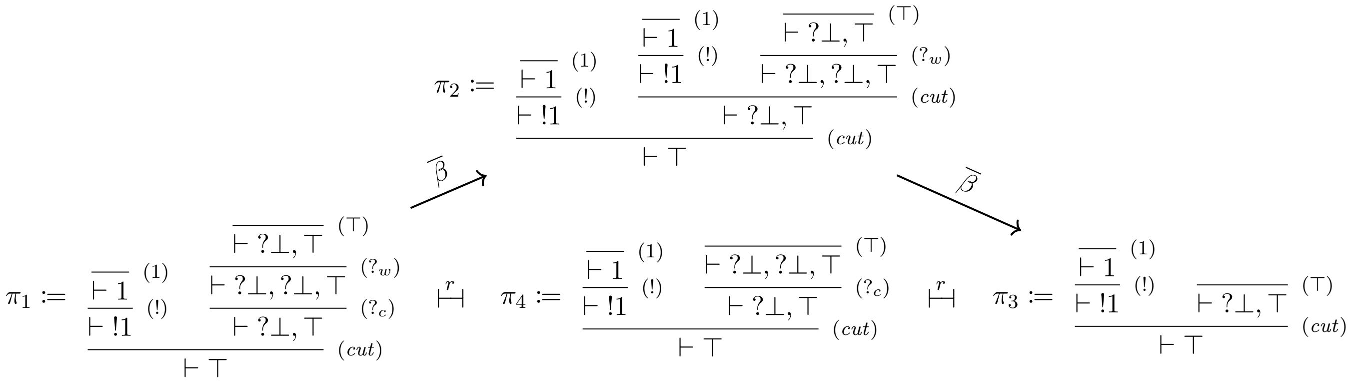

Figure 1: Counter-example to the strong normalization of $\xrightarrow{\overline{\beta}} \cdot \overset{r^*}{\vdash\!\dashv}$ where $\pi_1 \xrightarrow{\overline{\beta}} \pi_2 \xrightarrow{\overline{\beta}} \pi_3 \overset{r}{\vdash\!\dashv} \pi_4 \overset{r}{\vdash\!\dashv} \pi_1$

Derivations are often considered up to rule commutation, which is viewed as "bureaucracy" for all orders of rules are adequate but we cannot apply several rules at once. This explains why cut-elimination behaves better in proof-nets, that quotient derivations exactly by rule commutations [HG16]. This key result of confluence up to has so far only been proved for propositional multiplicative-additive linear logic [CP05, Theorem 5.1; DL25, Theorem 4.2], but was expected for the whole logic. We provide a proof of this result, showing that equality up to cut-elimination is simply rule commutations (between cut-free derivations). Equality up to rule commutation is simpler to study, and we show it is *undecidable* in propositional linear logic—and so is equality up to cut-elimination.

2 Second-order Linear Logic

Our framework is the standard sequent calculus of linear logic [Gir87] *with units and second-order quantifiers.* It is equipped with **cut-elimination** $\xrightarrow{\beta}$ and **rule commutation** $\overset{r}{\vdash\!\dashv}$. Some points of attention:

- we separate steps of cut-elimination as $\xrightarrow{\beta} = \xrightarrow{\overline{\beta}} \cup \overset{c}{\vdash\!\dashv}$ with the $cut - cut$ commutation $\overset{c}{\vdash\!\dashv}$ on one side, and all other rules $\xrightarrow{\overline{\beta}}$ on the other side (to obtain strong normalization);
- rule commutations $\overset{r}{\vdash\!\dashv}$ are the usual ones (see *e.g.* [Di24]), except there is no *cut*-rule above the rules that commute, and we do *not* have commutations with !-rules (not needed for confluence up to):[1]

$$\dfrac{\dfrac{\vdash A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad \overset{r}{\not\vdash\!\dashv} \quad \dfrac{\dfrac{\vdash A, ?B, ?B, ?\Gamma}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad\qquad \dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad \overset{r}{\not\vdash\!\dashv} \quad \dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

3 Rewriting theory modulo an equivalence relation

We assume known standard definitions and notations from rewriting theory [Ter03]: normal forms, $\xrightarrow{\alpha^+}$ is the transitive closure of $\xrightarrow{\alpha}$, etc. We will prove that cut-elimination is Church-Rosser modulo rule commutation.

Definition 1 (Church-Rosser modulo)**.** Let $\sim$ and $\to$ be relations on a set with $\sim$ an equivalence relation. We say $\to$ is **Church-Rosser modulo** $\sim$ when $(\to \cup \leftarrow \cup \sim)^* \subseteq \to^* \cdot \sim \cdot\ {}^*\!\leftarrow$.

The literature contains various results for proving Church-Rosser modulo, *e.g.* [JK84; Ohl98; AT12; Fel24]. Some are generalizations of Newman's Lemma [Ter03, Theorem 1.2.1] to prove Church-Rosser modulo instead of confluence. Unfortunately, most results do not apply to our case, in particular none of those from [Ter03, Section 14.3], with two due to Huet [Hue80, Lemmas 2.7, 2.8] and one to van Oostrom [Oos94,

[1] These rule commutations are exactly those given by the following algorithm. Consider a *cut*-rule with as premises two non *cut*-rules and such that two commutative cut-elimination cases can be applied on it: comparing the results of applying the left commutative step then the right, against applying the right then the left, yields a rule commutation (see the example of non-confluence in introduction and the proof of Proposition 8). Observe no commutation with a !-rule can be obtained this way.

Proposition 2.5.3]. First, closing diagrams with as hypotheses $a \overset{r^*}{\vdash\!\dashv} b$ is too complex, preventing from applying *e.g.* [Hue80, Lemma 2.7; Oos94, Proposition 2.5.3]. Second, strong normalization of $\overset{\beta}{\longrightarrow} \cdot \overset{r^*}{\vdash\!\dashv}$ is false—see Figure 1 for a counter-example—preventing from applying *e.g.* [Hue80, Lemma 2.8]. We use the following result, heavily inspired by a theorem from Aoto and Toyama [AT12, Theorem 2.2]. Intuitively, its item (ii) generalizes local confluence (*a.k.a.* property α) and its item (iii) local coherence (*a.k.a.* property γ).

Proposition 2. *Let $\rightarrow$, $\vdash\!\dashv$ and $\rightsquigarrow$ be relations on a set A such that $\vdash\!\dashv$ is symmetric and $\rightsquigarrow \subseteq \vdash\!\dashv$. Set $\Rightarrow := \rightarrow \cup \rightsquigarrow$. Suppose:*

(i) $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing;*

(ii) $\leftarrow \cdot \rightarrow \subseteq \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$*;*

(iii) *given $a, b, c \in A$, if $a \vdash\!\dashv b \rightarrow c$ then $a \rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow \cdot \leftarrow b$.*

Then $\rightarrow$ is Church-Rosser modulo $\overset{*}{\vdash\!\dashv}$.

Proof. The relation $(\rightsquigarrow^* \cdot \rightarrow \cdot \rightsquigarrow^*)^*$ is a well-founded strict partial order on A: transitivity is immediate, and irreflexivity and well-foundedness come from the strong normalization of $\rightarrow \cdot \rightsquigarrow^*$. For a sequence ρ of elements of A related by either $\vdash\!\dashv$, $\rightarrow$ or $\leftarrow$, pose the multiset $\mathcal{M}(\rho) := \biguplus_{a \vdash\!\dashv b \,\in\, \rho}\{a, a, b, b\} \uplus \biguplus_{a \rightarrow b \,\in\, \rho}\{a\} \uplus \biguplus_{a \leftarrow b \,\in\, \rho}\{b\}$, equipped with a well-founded strict partial order $<$ by the multiset extension of the one on A.

We proceed by induction on $\mathcal{M}(\rho)$. If ρ is of the shape $\rightarrow^* \cdot \overset{*}{\vdash\!\dashv} \cdot {}^*\!\leftarrow$ then we are done. Otherwise, ρ contains a sub-sequence $a \leftarrow b \rightarrow c$, or $a \vdash\!\dashv b \rightarrow c$, or $c \leftarrow b \vdash\!\dashv a$. If ρ contains a sub-sequence $a \leftarrow b \rightarrow c$, we replace it with $a \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow c$; the resulting sequence ρ' respects $\mathcal{M}(\rho') < \mathcal{M}(\rho)$, and we conclude by induction hypothesis. If ρ contains a sub-sequence $a \vdash\!\dashv b \rightarrow c$ or $c \leftarrow b \vdash\!\dashv a$, we replace $a \vdash\!\dashv b$ with $a \rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow \cdot \leftarrow b$, yielding a new sequence ρ' with $\mathcal{M}(\rho') < \mathcal{M}(\rho)$, and conclude by induction. ☐

4 Church-Rosser modulo for cut-elimination

We prove that cut-elimination $\overset{\beta}{\longrightarrow}$ is Church-Rosser modulo rule commutations $\overset{r^*}{\vdash\!\dashv}$, by showing that $\overset{\overline{\beta}}{\longrightarrow}$ is Church-Rosser modulo $(\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv})^*$ through Proposition 2. We instantiate the latter with $\rightarrow := \overset{\overline{\beta}}{\longrightarrow}$, $\vdash\!\dashv := \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}$ and $\rightsquigarrow := \overset{c}{\vdash\!\dashv} \cup \overset{r\backslash\top}{\vdash\!\dashv} \cup \overset{\top}{\longrightarrow}$, where $\overset{r}{\vdash\!\dashv} = \overset{r\backslash\top}{\vdash\!\dashv} \cup \overset{\top}{\longrightarrow} \cup \overset{\top}{\longleftarrow}$ is decomposed as such so as to prove Item (i) of Proposition 2. The relations $\overset{r\backslash\top}{\vdash\!\dashv}$ and $\overset{\top}{\longrightarrow}$ are defined (and justified) soon afterwards.

4.1 Strong normalization up to rule commutation

As shown on Figure 1, $\overset{\overline{\beta}}{\longrightarrow} \cdot \overset{r^*}{\vdash\!\dashv}$ is not strongly normalizing due to commutations with $\top$. Ergo, we *orient* those: note $\overset{r\backslash\top}{\vdash\!\dashv}$ the commutations without $\top$, and $\overset{\top}{\longrightarrow}$ those with $\top$ oriented in the "erasing" direction, *e.g.*:

$$\dfrac{\dfrac{}{\vdash A, B, \top, \Gamma}\,(\top)}{\vdash A \mathbin{⅋} B, \top, \Gamma}\,(⅋) \overset{\top}{\longrightarrow} \dfrac{}{\vdash A \mathbin{⅋} B, \top, \Gamma}\,(\top) \qquad \dfrac{\dfrac{}{\vdash \top, A[B/X], \Gamma}\,(\top)}{\vdash \top, \exists X A, \Gamma}\,(\exists) \overset{\top}{\longrightarrow} \dfrac{}{\vdash \top, \exists X A, \Gamma}\,(\top)$$

Note $\overset{r}{\vdash\!\dashv} = \overset{r\backslash\top}{\vdash\!\dashv} \cup \overset{\top}{\longrightarrow} \cup \overset{\top}{\longleftarrow}$. To apply Proposition 2, we need strong normalization of $\overset{\overline{\beta}}{\longrightarrow} \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\backslash\top}{\vdash\!\dashv} \cup \overset{\top}{\longrightarrow})^*$. This almost follows from a technical paper by Michele Pagani and Lorenzo Tortora de Falco [PT10] proving strong normalization of cut-elimination in proof-nets for second-order linear logic, using some non-standard proof-nets called *sliced pure structures*. The idea underlying our proof is a simple adaptation of [PT10]: there is a translation $\mathcal{T}$ from derivations of sequent calculus to sliced pure structures that satisfies the following.

- Images of $\mathcal{T}$ are easily shown to be strongly normalizing for a reduction procedure $\longrightarrow$.
- If $\pi \overset{\overline{\beta}}{\longrightarrow} \phi$ or $\pi \overset{\top}{\longrightarrow} \phi$ then $\mathcal{T}(\pi) \overset{*}{\longrightarrow} \mathcal{T}(\phi)$.
- If $\pi \overset{c}{\vdash\!\dashv} \phi$ or $\pi \overset{r\backslash\top}{\vdash\!\dashv} \phi$ then $\mathcal{T}(\pi) = \mathcal{T}(\phi)$.

Strong normalization for derivations then easily follows by studying a possible infinite reduction. Nonetheless, this is not a corollary of [PT10] but of a generalization of it with a few more reduction rules (corresponding to $\overset{\top}{\longrightarrow}$) and a slightly modified definition of sliced pure structures (to have invariance by $\top - \top$ commutation). Thence, we check that every following definition and statement in [PT10] still holds after these modifications.

Proposition 3. *The relation $\overset{\overline{\beta}}{\longrightarrow} \cdot (\overset{c}{\vdash\!\dashv} \cup \overset{r\backslash\top}{\vdash\!\dashv} \cup \overset{\top}{\longrightarrow})^*$ is strongly normalizing.*

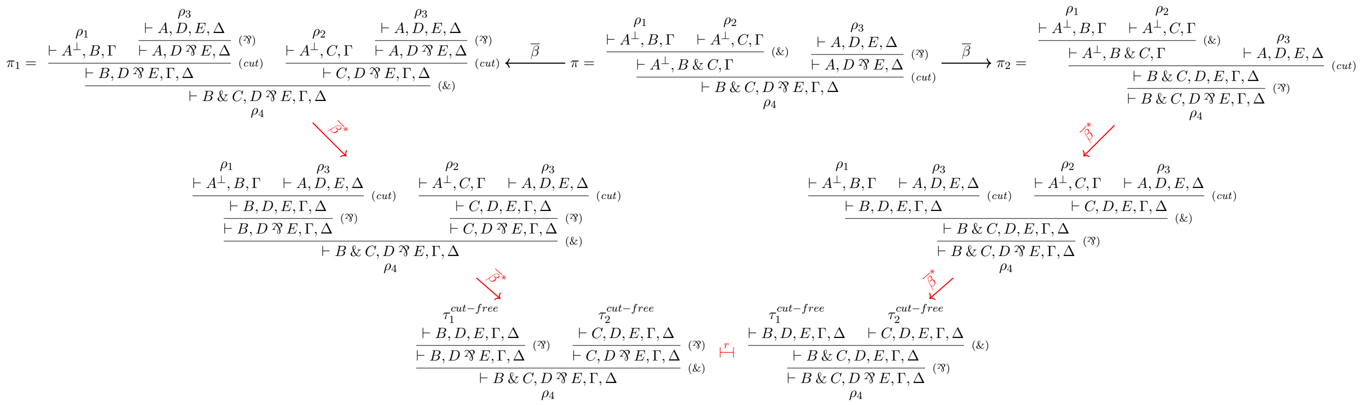

Figure 2: A critical pair for the proof of Lemma 4

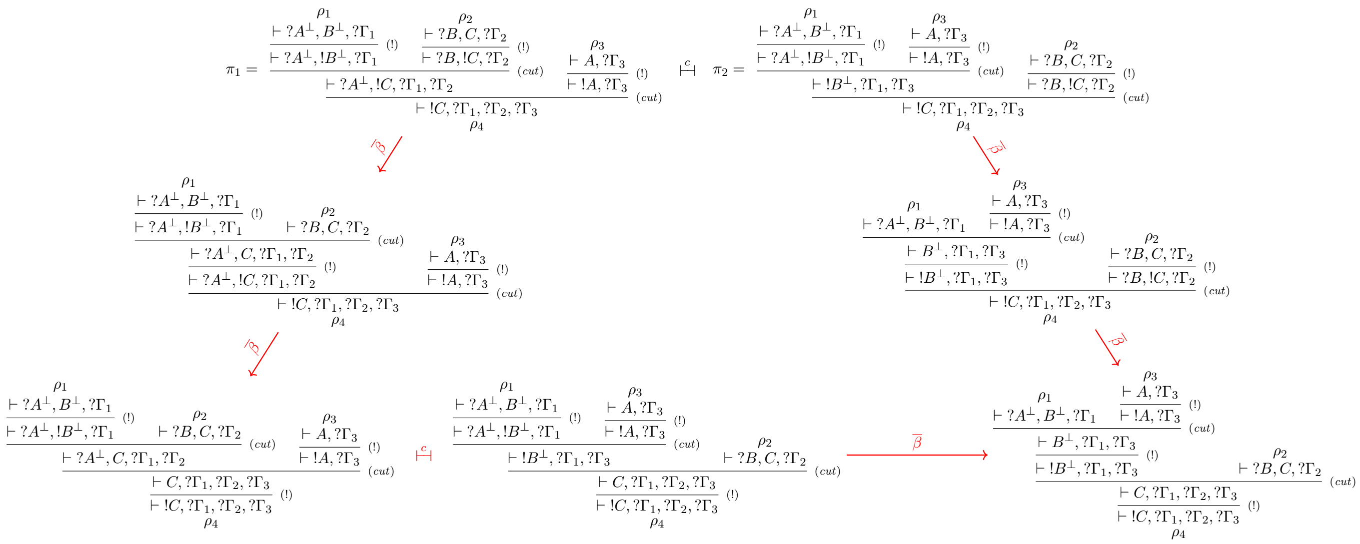

Figure 3: A critical pair for the proof of Lemma 6

4.2 Diagrams to close & Church-Rosser modulo

Items (ii) and (iii) of Proposition 2 with $\rightarrow := \overset{\beta}{\longrightarrow}$, $\vdash\!\dashv := \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}$ and $\rightsquigarrow := \overset{c}{\vdash\!\dashv} \cup \overset{r\setminus\top}{\vdash\!\dashv} \cup \overset{\top}{\longrightarrow}$ are given by:

Lemma 4. *If* $\pi_1 \overset{\beta}{\longleftarrow} \pi \overset{\beta}{\longrightarrow} \pi_2$, *then* $\pi_1 \overset{c^=}{\vdash\!\dashv} \cdot \overset{\beta^*}{\longrightarrow} \cdot (\overset{r\setminus\top^*}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv}) \cdot \overset{\beta^*}{\longleftarrow} \pi_2$.

Lemma 5. *If* $\pi_1 \overset{r}{\vdash\!\dashv} \pi_2 \overset{\beta}{\longrightarrow} \pi_3$, *then* $\pi_1 \overset{\beta^+}{\longrightarrow} \cdot \overset{\top^*}{\longrightarrow} \cdot \overset{r\setminus\top^*}{\vdash\!\dashv} \cdot \overset{\top^*}{\longleftarrow} \cdot \overset{\beta^+}{\longleftarrow} \pi_2$.

Lemma 6. *If* $\pi_1 \overset{c}{\vdash\!\dashv} \pi_2$, *then* $\pi_1 \overset{\beta^+}{\longrightarrow} \cdot \overset{c^*}{\vdash\!\dashv} \cdot \overset{\beta^*}{\longrightarrow} \cdot \overset{\beta^*}{\longleftarrow} \cdot \overset{c^*}{\vdash\!\dashv} \cdot \overset{\beta^+}{\longleftarrow} \pi_2$.

Proofs are case studies: a critical pair for Lemma 4 (resp. Lemma 6) is solved in Figure 2 (resp. Figure 3).

Proposition 7. *Cut-elimination* $\overset{\beta}{\longrightarrow}$ *is Church-Rosser modulo rule commutation* $\overset{r^*}{\vdash\!\dashv}$.

Proof. By Propositions 2 and 3 and Lemmas 4 to 6, $\overset{\beta}{\longrightarrow}$ is Church-Rosser modulo $\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv}$. Consider two derivations π and ϕ such that $\pi\ (\overset{\beta}{\longrightarrow} \cup \overset{\beta}{\longleftarrow} \cup \overset{r}{\vdash\!\dashv})^*\ \phi$. By Proposition 3, there exist cut-free derivations π' and ϕ' such that $\pi \overset{\beta^*}{\longrightarrow} \pi'$ and $\phi \overset{\beta^*}{\longrightarrow} \phi'$. Remark that $\pi'\ (\overset{\beta}{\longrightarrow} \cup \overset{\beta}{\longleftarrow} \cup \overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv})^*\ \phi'$. As $\overset{\beta}{\longrightarrow}$ is Church-Rosser modulo $\overset{r}{\vdash\!\dashv} \cup \overset{c}{\vdash\!\dashv}$, and π' and ϕ' are $\overset{\beta}{\longrightarrow}$-normal forms, it follows that $\pi'\ (\overset{c}{\vdash\!\dashv} \cup \overset{r}{\vdash\!\dashv})^*\ \phi'$. But no $\overset{c}{\vdash\!\dashv}$ step can be applied on cut-free derivations, and $\overset{r}{\vdash\!\dashv}$ preserves cut-freeness: $\pi' \overset{r^*}{\vdash\!\dashv} \phi'$, concluding the proof. ☐

5 Equality up to cut-elimination / up to rule commutation

We get our main result: two derivations are equal up to cut-elimination if and only if their normal forms are related by rule commutations. The direct way is Proposition 7, and the converse is simple but tedious.

Proposition 8 ($\overset{r}{\vdash\!\dashv} \subseteq =_\beta$)**.** *Given derivations π and τ, if $\pi \overset{r}{\vdash\!\dashv} \tau$ then $\pi \overset{\beta+}{\longleftarrow} \cdot \overset{\beta+}{\longrightarrow} \tau$.*

Proof. All cases are similar, we present only one (see it in the interactive prover C1ick ⅋ c⊗LLec⊥ here).

$$\dfrac{}{\vdash A \,\&\, B, \top, \Gamma}\,(\top) \overset{\beta+}{\longleftarrow} \dfrac{\dfrac{}{\vdash \top, 0}\,(\top) \qquad \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \quad \dfrac{}{\vdash B, \top, \Gamma}\,(\top)}{\vdash A \,\&\, B, \top, \Gamma}\,(\&)}{\vdash A \,\&\, B, \top, \Gamma}\,(cut) \overset{\beta+}{\longrightarrow} \dfrac{\dfrac{}{\vdash A, \top, \Gamma}\,(\top) \quad \dfrac{}{\vdash B, \top, \Gamma}\,(\top)}{\vdash A \,\&\, B, \top, \Gamma}\,(\&)$$

□

Theorem 9. *Two derivations are equal up to cut-elimination $\overset{\beta}{\longrightarrow}$ if and only if any of their $\overset{\beta}{\longrightarrow}$-normal forms are related by rule commutations $\overset{r^*}{\vdash\!\dashv}$.*

Proof. Consider derivations π and ϕ with respective $\overset{\beta}{\longrightarrow}$-normal forms π' and ϕ'. Note $\pi =_\beta \phi \iff \pi' =_\beta \phi'$. If $\pi' =_\beta \phi'$ then $\pi' \overset{r^*}{\vdash\!\dashv} \phi'$ (Proposition 7 on cut-free derivations). If $\pi' \overset{r^*}{\vdash\!\dashv} \phi'$ then $\pi' =_\beta \phi'$ (Proposition 8). □

Our proof of Theorem 9 applies to *most* fragments of second-order linear logic: only Proposition 8 (and then Theorem 9) needs some hypotheses for the exponentials.

Theorem 10. *Two derivations are equal up to cut-elimination $\overset{\beta}{\longrightarrow}$ if and only if any of their $\overset{\beta}{\longrightarrow}$-normal forms are related by rule commutations $\overset{r^*}{\vdash\!\dashv}$, in all fragments of linear logic such that:*

- *if the $?_c$-rule belongs to the fragment, then the ⅋- and $\otimes$-rules also belong to it;*
- *if the $?_w$-rule belongs to the fragment, then the $\bot$- and 1-rules also belong to it.*

6 Undecidability of equality up to rule commutation/cut-elimination

We now consider only propositional linear logic, *without quantifiers*. We examine three problems:

Equality up to Cut: Given two derivations, are they equal up to cut-elimination?

Proof Equivalence: Given two cut-free derivations, are they equal up to rule commutation?

Provability: Given a sequent, is it provable?

By Theorem 9, Equality up to Cut is undecidable if and only if Proof Equivalence is. We reduce to the later Provability, well-known to be undecidable [Lin95].

Lemma 11. *For any formula A,* $\left(\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r^*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_2) \right) \iff$ *A is provable*

Proof. If A is provable, then we can exhibit rule commutations linking these two derivations. If A is not provable, then we can compute the equivalence classes of both derivations and check they are disjoint. □

Proposition 12. *Proof Equivalence and Equality up to Cut are undecidable in propositional linear logic.*

Proof. Provability is undecidable [Lin95]. By Lemma 11, this entails undecidability of Proof Equivalence, which entails by Theorem 9 undecidability of Equality up to Cut (actually, we cannot even decide whether two cut-free derivations are equal up to cut-elimination). □

7 Conclusion

Cut-elimination in second-order linear logic is Church-Rosser modulo rule commutation, a result that was widely expected but not yet proved. Hence, equality up to cut-elimination coincides with rule commutations on cut-free derivations. This holds in most sub-systems of linear logic, *e.g.* multiplicative-exponential linear logic. Also, adding for instance the *mix*-rules is easy. Furthermore, we can extend this in presence of other usual transformations of derivations, such as axiom-expansion.

We establish equality up to rule commutation is undecidable in propositional linear logic. This completely prevents any canonical notion of proof-nets for propositional linear logic, and adding the *mix*-rule does not solve the problem for our proof easily extends to that setting. This contrasts sharply with the case of multiplicative linear logic where adding the *mix*-rules turns the problem from PSPACE-complete [HH16] to solvable by proof-nets—see [FR94] for the first definition of these proof-nets.

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