No Uniform Interpolation in Linear Logic

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Abstract

The literature on uniform interpolation in linear logic is very scarce: the sole known result is that the multiplicative-additive fragment of linear logic has the uniform interpolation property, and so does its intuitionistic and affine variants [1]. We exhibit a simple formula of the unit-free multiplicative-exponential fragment of linear logic that cannot have any uniform interpolant in propositional linear logic. This example shows not only that propositional linear logic does not have the uniform interpolation property, but also that its intuitionistic and affine variants neither do.

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1 Introduction

Looking at known results for interpolation in linear logic and its fragments, a reader can find a modicum of work, chiefly on Craig's and Lyndon's interpolation and with very little on uniform interpolation. *Craig's interpolation* states that if $A \vdash B$ is provable, then there exists a formula C built only from variables occurring in both A and B and such that both $A \vdash C$ and $C \vdash B$ are provable. *Lyndon's interpolation* is a strengthening of this property taking account the polarity of variables. These two properties have been proved in first-order linear logic and studied in several fragments of this logic. Notably, Roorda [19] proved and disproved Lyndon's interpolation in various fragments of propositional linear logic; Brotherston and Rajeev [2] used display logic to obtain Craig's interpolation in the multiplicative and multiplicative-additive fragments of linear logic, as well as in classical logic (they then formalised these results working with Dawson [5]); Strassburger [22] provides in the calculus of structures for multiplicative-exponential linear logic a result corresponding to Craig's interpolation; a proof of Craig's and Lyndon's interpolation in first-order linear logic, along with links to cut-elimination, has been developed by Saurin [20], who then considered with Fiorillo and Osorio-Valencia [7] this question in the setting of the proof-net syntax of multiplicative linear logic. Let us also mention that Fussner and Santschi [8] researched Craig's interpolation in extensions of classical and intuitionistic propositional linear logic using algebraic techniques.

The literature is even more scarce on uniform interpolation in linear logic. *Uniform interpolation*, introduced by Pitts [18] in the setting of intuitionistic logic, is a stronger version of Craig's interpolation where the interpolant depends only on one of the two formulas under consideration. More precisely, a uniform interpolant for a formula A, with respect to a variable X, is a formula C such that:

- the variables of C are included in those of A, except X must not appear in C;
- $A \vdash C$ is provable;
- for all formulas B such that X does not appear in B and $A \vdash B$ is provable, $C \vdash B$ is provable too.

In particular, uniform interpolants are an interpretation of first-order quantifiers in the logic under consideration, since the formula C above behaves like $\exists X A$ (at least for provability purposes). Not every logic has uniform interpolation, for instance the modal logic S4 does not [10] even thought it has Craig's and Lyndon's interpolations [17, 9]. It is easy to check that classical logic has this property: a uniform interpolant for a formula A with respect to X is simply $A[\top/X] \vee A[\bot/X]$, that is the disjunction of substituting X by true or false. The foundational work of Pitts [18] proves that intuitionistic logic has uniform interpolation. The algorithm to compute an interpolant is more complex than in the classical case, and the proof technique is based on a syntactic approach. This method by Pitts was then adapted to many other logics. In particular, it was used by Alizadeh, Derakhshan and Ono [1] to prove uniform interpolation in the Lambek calculus, as well as in classical and intuitionistic multiplicative-additive linear logic – and also in the affine variants of these two. This is, at the best of the author's knowledge, the sole result on uniform interpolation in linear logic to this day. Pitts' method does not extend easily to full propositional linear logic because of the contraction rule, that has a premise sequent "bigger" than its conclusion sequent, and that necessitated a particular treatment in Pitts' paper [18] so as to remove it from the calculus.

Nonetheless, since classical logic and intuitionistic logic both have uniform interpolation, one could expect linear logic to have it too. We show it is not the case by exhibiting a counter-example. Actually, we present formulas A and $(B_n)_{n\in\mathbb{N}}$ such that:

1. an atom X appears in A but not in any of the B_n;
2. $A \vdash B_n$ is provable for all number n;
3. for all formulas C such that X does not appear in C, for any number $n \in \mathbb{N}$, if $A \vdash C$ and $C \vdash B_n$ are both provable, then C is of size at least n.

The result immediately follows: A has no uniform interpolant with respect to X, because such an interpolant would have an unbounded size. Furthermore, the counter-example we exhibit is not only one for uniform interpolation in linear logic, but also in its intuitionistic and affine variants. A subtle use of the exponential modality is mandatory inside the formula A, since uniform interpolation holds with no exponential (case of multiplicative-additive linear logic) and with exponentials everywhere (case of classical logic). This complexity of A entails difficulties to prove some results for sequents of the shape $A \vdash C$. In particular, the most challenging part in this paper is the proof of property 3, that involves some technical tools among which geometry of interaction.

Outline. We first recall the syntax of linear logic in Section 2, along with a definition of uniform interpolation in this setting. Then, we define some useful but technical tools in Section 3. The main part, Section 4, consists of a presentation of our counter-example and a proof it cannot have a uniform interpolant using the tools from the preceding section. Finally, we show in Section 5 that our counter-example is also one for uniform interpolation in *intuitionistic* linear logic and in *affine* logic (both classical and intuitionistic).

2 Linear Logic & Uniform Interpolation

We use the standard presentation of linear logic as a unilateral sequent calculus [11]. In particular, a proof is a *derivation* whose conclusion sequent is of the shape $\vdash \Gamma$.

$$\frac{}{\vdash X^+, X^-}\ (ax) \qquad \frac{\vdash A, \Gamma \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

Figure 1 Rules of propositional Linear Logic

Formulas are defined by the following grammar, with X an atom in some countable set:

$$A, B ::= \underbrace{X^- \mid X^+}_{\text{atom}} \mid \underbrace{A \parr B \mid A \otimes B \mid \bot \mid 1}_{\text{multiplicative}} \mid \underbrace{A \,\&\, B \mid A \oplus B \mid \top \mid 0}_{\text{additive}} \mid \underbrace{?A \mid !A}_{\text{exponential}}$$

Orthogonality $(\cdot)^\perp$ (*a.k.a.* negation, duality) is an involution defined inductively by:

$$\begin{array}{llll}
(X^-)^\perp \overset{\text{def}}{=} X^+ & (A \parr B)^\perp \overset{\text{def}}{=} A^\perp \otimes B^\perp & (A \,\&\, B)^\perp \overset{\text{def}}{=} A^\perp \oplus B^\perp & (?A)^\perp \overset{\text{def}}{=} !A^\perp \\
(X^+)^\perp \overset{\text{def}}{=} X^- & (A \otimes B)^\perp \overset{\text{def}}{=} A^\perp \parr B^\perp & (A \oplus B)^\perp \overset{\text{def}}{=} A^\perp \,\&\, B^\perp & (!A)^\perp \overset{\text{def}}{=} ?A^\perp \\
 & \bot^\perp \overset{\text{def}}{=} 1 & \top^\perp \overset{\text{def}}{=} 0 & \\
 & 1^\perp \overset{\text{def}}{=} \bot & 0^\perp \overset{\text{def}}{=} \top &
\end{array}$$

Linear implication is defined as $A \multimap B \overset{\text{def}}{=} A^\perp \parr B$. **Rules** are given on Figure 1, where A and B stand for arbitrary formulas, Γ and Δ for sets of (occurrences of) formulas, and $?\Gamma$ for a set of (occurrences of) ?-formulas. Remark we restrict the *ax*-rule to *atomic formulas*, a well-known simplification. This sequent calculus is equipped with a **cut-elimination** procedure, denoted by $\longrightarrow$, that is weakly normalising and that we recall in Appendix A.

Uniform interpolation in linear logic can be stated as follows. We denote by $\mathcal{V}(A)$ the set of atoms occurring in the formula A – *e.g.* $\mathcal{V}((X^+ \parr X^-) \,\&\, Y^-) = \{X, Y\}$.

▶ **Definition 1** (Uniform Interpolant). *Given a formula A and an atom X, a **uniform interpolant** for A with respect to X is a formula C such that:*

- $\mathcal{V}(C) \subseteq \mathcal{V}(A) \setminus \{X\}$*;*
- $\vdash A^\perp, C$ *is provable;*
- *for all formulas B, if $X \notin \mathcal{V}(B)$ and $\vdash A^\perp, B$ is provable then $\vdash C^\perp, B$ is provable.*

*The **uniform interpolation property** holds when every formula admits a uniform interpolant with respect to every atom.*

3 Toolbox

We present here two tools used to prove that our counter-example has no uniform interpolant. The first is a variant of *slices*, the second of *Geometry Of Interaction*. They are needed solely to prove our most technical statement, corresponding to property 3 in the introduction.

3.1 A variant of Slices

The &-rule stands apart in linear logic since it is the sole rule with some sharing of the context Γ. A useful tool when considering &-rules is the notion of a *slice* [11, 13] which is a derivation missing some additive component: a slice is obtained from a derivation by deleting one sub-tree of each &-rule. In a slice, no rule duplicates a context. As our framework has not only additive but also exponential rules, the well-behaved concept is the one of a 0-*slice*, meaning we only slice &-rules that are not above a !-rule, *i.e.* those at depth 0.

▶ **Definition 2** (0-slice). *For π a derivation, a 0-**slice** of π is a (usually non-derivation) tree π' obtained by deleting one of the two sub-trees of each &-rule of π* that is not above a !-rule. *Thus, in π' some (but maybe not all) &-rules are unary:*

$$\dfrac{\vdash A, \Gamma}{\vdash A \& B, \Gamma}\,(\&_1) \qquad \dfrac{\vdash B, \Gamma}{\vdash A \& B, \Gamma}\,(\&_2)$$

▶ **Example 3.** Here is a derivation followed by its two 0-slices (note it has three slices):

$$\dfrac{\dfrac{}{\vdash \top, !(1 \& 1)}\,(\top) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1 \& 1}\,(\&)}{\vdash !(1 \& 1)}\,(!)}{\vdash \bot, !(1 \& 1)}\,(\bot)}{\vdash \top \& \bot, !(1 \& 1)}\,(\&) \qquad \dfrac{\dfrac{}{\vdash \top, !(1 \& 1)}\,(\top)}{\vdash \top \& \bot, !(1 \& 1)}\,(\&_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1 \& 1}\,(\&)}{\vdash !(1 \& 1)}\,(!)}{\vdash \bot, !(1 \& 1)}\,(\bot)}{\vdash \top \& \bot, !(1 \& 1)}\,(\&_2)$$

Cut-elimination can be defined in the natural way for 0-slices, taking the definition for derivations and extending the $\& - \oplus$ key case with

$$\dfrac{\dfrac{\vdash A, \Gamma}{\vdash A \& B, \Gamma}\,(\&_1) \quad \dfrac{\vdash A^\perp, \Delta}{\vdash A^\perp \oplus B^\perp, \Delta}\,(\oplus_1)}{\vdash \Gamma, \Delta}\,(cut) \quad \longrightarrow \quad \dfrac{\vdash A, \Gamma \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,(cut)$$

$$\dfrac{\dfrac{\vdash B, \Gamma}{\vdash A \& B, \Gamma}\,(\&_2) \quad \dfrac{\vdash B^\perp, \Delta}{\vdash A^\perp \oplus B^\perp, \Delta}\,(\oplus_2)}{\vdash \Gamma, \Delta}\,(cut) \quad \longrightarrow \quad \dfrac{\vdash B, \Gamma \quad \vdash B^\perp, \Delta}{\vdash \Gamma, \Delta}\,(cut)$$

as well as replacing &-rules with $\&_1$- or $\&_2$-rules during a $?_d - !$ key case (where a !-rule is removed). Observe it is not possible to eliminate *cut*-rules of the two following shapes:

$$\dfrac{\dfrac{\vdash A, \Gamma}{\vdash A \& B, \Gamma}\,(\&_1) \quad \dfrac{\vdash B^\perp, \Delta}{\vdash A^\perp \oplus B^\perp, \Delta}\,(\oplus_2)}{\vdash \Gamma, \Delta}\,(cut) \qquad \dfrac{\dfrac{\vdash B, \Gamma}{\vdash A \& B, \Gamma}\,(\&_2) \quad \dfrac{\vdash A^\perp, \Delta}{\vdash A^\perp \oplus B^\perp, \Delta}\,(\oplus_1)}{\vdash \Gamma, \Delta}\,(cut)$$

The only property of 0-slices we need is their behaviour with respect to cut-elimination.

▶ **Lemma 4.** *Let π and ϕ be derivations such that $\pi \longrightarrow \phi$. For each 0-slice ϕ' of ϕ, there exists a 0-slice π' of π such that $\pi' \longrightarrow \phi'$ or $\pi' = \phi'$.*

Proof. We can check that each cut-elimination step respects this property, with the equality case coming from a reduction on rules not in the considered 0-slice. ◀

▶ Remark 5. Observe a corresponding result for slices does not hold, because a formula of the shape $?(A \oplus B)$ can be associated to both a $\oplus_1$- and a $\oplus_2$-rule. This is a phenomenon at play, for instance, in the Seely isomorphism $?(A \oplus B) \simeq ?A \parr ?B$. The cut-elimination case responsible for this failure is the $?_c - !$ key case, because the two copies of the duplicated derivation (the "! box") may not delete the same sub-trees for all of their corresponding &-rules.

3.2 A variant of Geometry Of Interaction

We define here the key tool for proving our counter-example has no uniform interpolant, previously used in the restricted setting of unit-free multiplicative-additive linear logic [6]. It falls within the spirit of *Geometry Of Interaction* (shortened as *GOI*) [12]. The quite unusual part is that in the next section we will look at *connectives* of the *cut*-formulas along a path from GOI.

▶ **Definition 6** (GOI projection graph). *The **GOI projection graph** $\mathcal{G}_\pi$ of a derivation, or of a* 0*-slice, π of conclusion $\vdash \Gamma$ is the undirected edge-bicoloured graph defined as follows, colors being ax and cut.*

Vertices: *All occurrences of atomic formulas X^+ and X^- (for all atoms) from the formulas of Γ and from the cut-formulas A and $A^\perp$ in π, with each cut-rule yielding a pair of cut-formulas giving separate and dual occurrences.*

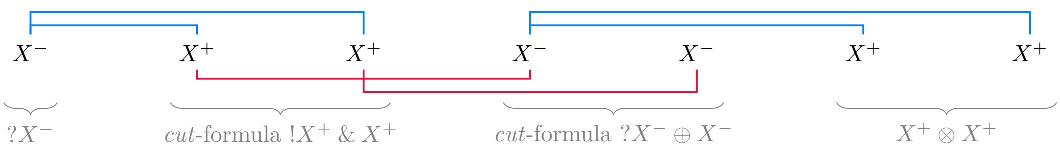
ax-edges: *For each pair of occurrences X^+ and X^- such that π contains an ax-rule $\dfrac{}{\vdash X^+, X^-}\,(ax)$, there is an ax-edge between X^+ and X^-.*

cut-edges: *For each cut-rule $\dfrac{\vdash A, \Delta \qquad \vdash A^\perp, \Sigma}{\vdash \Delta, \Sigma}\,(cut)$ in π, there is a cut-edge between dual occurrences from the two formulas A and $A^\perp$.*

In our figures, ax-edges are coloured in **a**zure and depicted **a**bove the vertices, whereas cut-edges are coloured in **c**rimson and depicted below the vertices.

▶ **Example 7.** Here is a derivation π followed by its GOI projection graph $\mathcal{G}_\pi$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^-, X^+$}
\RightLabel{$(?_d)$}
\UnaryInfC{$\vdash ?X^-, X^+$}
\RightLabel{$(!)$}
\UnaryInfC{$\vdash ?X^-, !X^+$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^-, X^+$}
\RightLabel{$(?_d)$}
\UnaryInfC{$\vdash ?X^-, X^+$}
\RightLabel{$(\&)$}
\BinaryInfC{$\vdash ?X^-, !X^+ \,\&\, X^+$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^-, X^+$}
\RightLabel{$(?_d)$}
\UnaryInfC{$\vdash ?X^-, X^+$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^-, X^+$}
\RightLabel{$(?_d)$}
\UnaryInfC{$\vdash ?X^-, X^+$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash ?X^-, ?X^-, X^+ \otimes X^+$}
\RightLabel{$(?_c)$}
\UnaryInfC{$\vdash ?X^-, X^+ \otimes X^+$}
\RightLabel{$(\oplus_1)$}
\UnaryInfC{$\vdash ?X^- \oplus X^-, X^+ \otimes X^+$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash ?X^-, X^+ \otimes X^+$}
\end{prooftree}
$$



▶ Remark 8. Observe they are two ways to obtain several ax-edges on a same occurrence: this occurrence can be duplicated using a $?_c$-rule on one of its ancestor (in the formula tree), or by being in the context of a &-rule. Both cases can be observed in Example 7.

We will consider **alternating** paths in a GOI projection graph, which are paths[1] where consecutive edges are coloured differently – *e.g.* we start with an ax-edge, then take a cut-edge, then an ax-edge, and so on. We have a typical result of GOI about *persistent* paths (see *e.g.* [3]): some paths are preserved through cut-elimination. What is relevant for our purposes is that this holds not only for cut-elimination between derivations but also between 0-slices.

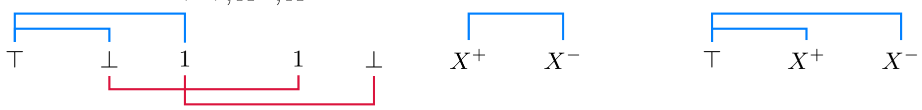
[1] A path is a finite alternating sequence of vertices and edges $(v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n)$ such that for all $i \in \{1, \ldots, n\}$, the two endpoints of e_i are v_{i-1} and v_i.

▶ **Lemma 9.** *Consider a cut-elimination step $\pi \longrightarrow \phi$ between two derivations, or between two 0-slices, of conclusion $\vdash \Gamma$. Let u and v be occurrences of atoms from Γ. If there is an alternating path between u and v in $\mathcal{G}_\phi$, then there is an alternating path between them in $\mathcal{G}_\pi$.*

Proof. By inspection on the kind of the cut-elimination step. Since we only care about ax-rules and atoms, the non-immediate cases are either trivial with $\mathcal{G}_\phi \subseteq \mathcal{G}_\pi$ ($\& - \oplus$ key cases, $?_d - !$ key case, $?_w - !$ key case, $\top - cut$ commutative case) or easy to check (ax key case, $?_c - !$ key case, $\& - cut$ commutative case). ◀

▶ Remark 10. It is easy to find a counter-example to the reciprocal of Lemma 9. One could be tempted to consider not only ax-rules but also $\top$-rules in a GOI projection graph, also adding vertices for all occurrences of units. Such an altered setting does not suit our purposes for Lemma 9 would not hold – consider the following, with the occurrences of $\top$ and X^+:

$$\dfrac{\dfrac{}{\vdash \top, \bot \parr 1}(\top) \qquad \dfrac{\dfrac{}{\vdash 1}(1) \qquad \dfrac{\dfrac{}{\vdash X^+, X^-}(ax)}{\vdash \bot, X^+, X^-}(\bot)}{\vdash 1 \otimes \bot, X^+, X^-}(\otimes)}{\vdash \top, X^+, X^-}(cut) \qquad \longrightarrow \qquad \dfrac{}{\vdash \top, X^+, X^-}(\top)$$



4 A Counter-Example to Uniform Interpolation

Let us fix two atoms X and Y. Our counter-example is:

$$A \overset{\text{def}}{=} X^+ \otimes !\big(X^+ \multimap (X^+ \otimes Y^+)\big) \otimes (X^+ \multimap Y^+) \qquad \begin{cases} B_0 & \overset{\text{def}}{=} Y^+ \otimes Y^+ \\ B_{n+1} & \overset{\text{def}}{=} B_n \otimes Y^+ \end{cases} \tag{1}$$

The main intuition is the following. The formula A is made of three parts: X^+, a "reusable machine" taking as input a X^+ and returning a X^+ and a Y^+, and a "one-use machine" taking as input a X^+ and returning a Y^+. Thence, A can "produce" any non-null number of Y^+, so that $A \vdash B_n$ is provable for all $n \in \mathbb{N}$. Therefore, one may think that $!Y^+ \otimes Y^+$ is a uniform interpolant for A, and indeed we do have $!Y^+ \otimes Y^+ \vdash B_n$ provable; however, $A \vdash !Y^+ \otimes Y^+$ cannot be proved. This generalizes to all "uniform interpolant candidates".

Let us point out that A can be viewed as some "pseudo ! exponential" of Y^+, since the following dereliction and selection rules for Y^- are admissible:

$$\dfrac{\vdash Y^-, \Gamma}{\vdash A^\perp, \Gamma} \qquad\qquad \dfrac{\vdash Y^-, A^\perp, \Gamma}{\vdash A^\perp, \Gamma}$$

It is reminiscent of two other "pseudo ! exponential" of Y^+ from the literature:

- $Y^+ \otimes !(Y^+ \multimap (Y^+ \otimes Y^+)) \otimes !(Y^+ \multimap 1)$ used by Yves Lafont to discuss some non-admissible !-introduction rule [15, page 14];
- $X^+ \otimes !(X^+ \multimap (X^+ \otimes X^+)) \otimes !(X^+ \multimap 1) \otimes !(X^+ \multimap Y^+)$ (actually a formula isomorphic to it) used by Anupam Das to prove different the exponentials of linear logic and their encoding in multiplicative-additive linear logic with least and greatest fixed points [4].

The later formula by Anupam Das could also be used here in place of our A. Nonetheless, we preferred the definition in Equation (1) because it is smaller and we do not need the full power of an exponential – in particular not the $?w$-rule.

To obtain that there is no uniform interpolant for A, it stays to check the three properties claimed in the introduction: $X \notin \mathcal{V}(B_n)$; and $A \vdash B_n$ is provable; and for all formulas C such that $X \notin \mathcal{V}(C)$, if $A \vdash C$ and $C \vdash B_n$ are both provable, then C is of size at least n. Note that the first property holds by definition. The second is easy to prove.

▶ **Lemma 11.** *For all $n \in \mathbb{N}$, there is a derivation of $\vdash A^\perp, B_n$.*

Proof. We build a derivation π_n of $\vdash X^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n$ by induction on $n \in \mathbb{N}$. The result then follows by exhibiting:

$$\dfrac{\dfrac{\overset{\pi_n}{\vdash X^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n}}{\vdash X^-, ?\,(X^+ \otimes (X^- \parr Y^-)) \parr (X^+ \otimes Y^-), B_n}\,(\parr)}{\vdash X^- \parr ?\,(X^+ \otimes (X^- \parr Y^-)) \parr (X^+ \otimes Y^-), B_n}\,(\parr)$$

We define the wanted derivations as:

$$\begin{cases}
\pi_0 \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax) \qquad \dfrac{\dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax) \qquad \dfrac{\dfrac{}{\vdash Y^-, Y^+}\,(ax) \qquad \dfrac{}{\vdash Y^-, Y^+}\,(ax)}{\vdash Y^-, Y^-, Y^+ \otimes Y^+}\,(\otimes)}{\vdash X^-, Y^-, X^+ \otimes Y^-, Y^+ \otimes Y^+}\,(\otimes)}{\vdash X^- \parr Y^-, X^+ \otimes Y^-, Y^+ \otimes Y^+}\,(\parr)}{\vdash X^-, X^+ \otimes (X^- \parr Y^-), X^+ \otimes Y^-, Y^+ \otimes Y^+}\,(\otimes)}{\vdash X^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, Y^+ \otimes Y^+}\,(?_d) \\[2em]
\pi_{n+1} \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax) \qquad \dfrac{\dfrac{\overset{\pi_n}{\vdash ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n} \qquad \dfrac{}{\vdash Y^-, Y^+}\,(ax)}{\vdash X^-, Y^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n \otimes Y^+}\,(\otimes)}{\vdash X^- \parr Y^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n \otimes Y^+}\,(\parr)}{\vdash X^-, X^+ \otimes (X^- \parr Y^-), ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n \otimes Y^+}\,(\otimes)}{\vdash X^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n \otimes Y^+}\,(?_d)}{\vdash X^-, ?\,(X^+ \otimes (X^- \parr Y^-))\,, X^+ \otimes Y^-, B_n \otimes Y^+}\,(?_c)
\end{cases}$$

◀

The third wished property is more complex to prove: we use the tools from Section 3. It also needs an intermediate result, that is not surprising, albeit a little knotty to prove.

▶ **Lemma 12.** *Let C be a formula such that $X \notin \mathcal{V}(C)$ and $\vdash A^\perp, C$ is provable. There exists a cut-free derivation of $\vdash A^\perp, C$ in which no $!$-rule has $?\,(X^+ \otimes (X^- \parr Y^-))$ in its conclusion sequent.*

Proof. Consider a cut-free derivation π of $\vdash A^\perp, C$ *with a minimal number of $!$-rules having $?\,(X^+ \otimes (X^- \parr Y^-))$ in their conclusion sequents.* We prove this number is null.

By the sub-formula property [11] and the constraint on the context in a $!$-rule, any given $!$-rule in π is of the shape

$$\dfrac{\overset{\phi}{\vdash \left(?\,(X^+ \otimes (X^- \parr Y^-))\right)^i, D, ?\Delta}}{\vdash \left(?\,(X^+ \otimes (X^- \parr Y^-))\right)^i, !D, ?\Delta}\,(!)$$

with a conclusion sequent containing $i \geq 0$ copies of $?\,(X^+ \otimes (X^- \parr Y^-))$ for some $i \in \mathbb{N}$, and where $!D, ?\Delta$ are sub-formulas of C. Towards a contradiction, assume $i \geq 1$. Substituting

X by 0 in ϕ yields a derivation $\phi[0/X]$ of $\vdash (?\,(0 \otimes (\top \parr Y^-)))^i, D, ?\Delta$ since, by hypothesis, there is no X in $D, ?\Delta$. Consider any result τ of fully eliminating *cut*-rules in

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \top, 0 \otimes Y^+}\,(\top)}{\vdash \top \parr (0 \otimes Y^+)}\,(\parr)}{\vdash\, !(\top \parr (0 \otimes Y^+))}\,(!) \qquad \dfrac{\dfrac{\phi[0/X]}{\vdash (?\,(0 \otimes (\top \parr Y^-)))^i, D, ?\Delta}}{\vdash ?\,(0 \otimes (\top \parr Y^-)), D, ?\Delta}\,(?_c)}{\dfrac{\vdash D, ?\Delta}{\vdash !D, ?\Delta}\,(!)}\,(cut)}{\vdash (?\,(X^+ \otimes (X^- \parr Y^-)))^i, !D, ?\Delta}\,(?_w)$$

where doubled inference rules mean we apply the rule several times depending on i. Then, τ has no !-rule having $?\,(X^+ \otimes (X^- \parr Y^-))$ in its conclusion sequent by the sub-formula property. Therefore, the derivation obtained from π by substituting with τ the sub-tree $\dfrac{\dfrac{\phi}{\vdash (?\,(X^+ \otimes (X^- \parr Y^-)))^i, D, ?\Delta}}{\vdash (?\,(X^+ \otimes (X^- \parr Y^-)))^i, !D, ?\Delta}\,(!)$ contradicts the minimality assumption on π. ◀

▶ **Proposition 13.** *Let C be a formula such that $X \notin \mathcal{V}(C)$. If $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$ are both provable for some number $n \in \mathbb{N}$, then C contains at least $n + 2$ occurrences of Y^+.*

Proof. Let us call π and ϕ derivations of respectively $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$. We assume them *cut*-free without any loss of generality. We will use the GOI projection graph of (some 0-slice of) their composition by *cut* to identify Y^-'s in $C^\perp$ linked to the Y^+'s of B_n. Thanks to alternating paths, we will prove that these occurrences of Y^- are pairwise distinct. This yields one Y^- in $C^\perp$ for each Y^+ in B_n, concluding the proof.

Finding an alternating path in some 0-slice. Observe that $\dfrac{\dfrac{\pi}{\vdash A^\perp, C} \quad \dfrac{\phi}{\vdash C^\perp, B_n}}{\vdash A^\perp, B_n}\,(cut)$ reduces to a *cut*-free derivation of $\vdash A^\perp, B_n$, that we call τ. This τ is its own unique 0-slice, since there is no &-connective in $A^\perp$ nor in B_n. By repeated applications of Lemma 4, there is a 0-slice μ of $\dfrac{\dfrac{\pi}{\vdash A^\perp, C} \quad \dfrac{\phi}{\vdash C^\perp, B_n}}{\vdash A^\perp, B_n}\,(cut)$ such that $\mu \longrightarrow^* \tau$.

Consider some occurrence Y^+_α in B_n, with the index α used to distinguish it from other occurrences. Since τ is a *cut*-free derivation of $\vdash A^\perp, B_n$, there must be an *ax*-rule between this occurrence Y^+_α and one of the two occurrences of Y^- in $A^\perp$: call it Y^-_1. Hence, there is an alternating path (made of one *ax*-edge) between these two occurrences in the GOI projection graph $\mathcal{G}_\tau$ of τ. By repeated applications of Lemma 9, there also is an alternating path p between Y^-_1 and Y^+_α in $\mathcal{G}_\mu$. Thus, there is an occurrence Y^-_β and an *ax*-rule $\dfrac{}{\vdash Y^-_\beta, Y^+_\alpha}\,(ax)$ in μ and Y^-_β–Y^+_α is the last edge of p. This occurrence Y^-_β can only be in $C^\perp$.

We use the alternating path p to show there is no sub-formula $?D$ of $C^\perp$ containing Y^-_β – say differently, there is no ? between the root of the formula $C^\perp$ and Y^-_β. Once this claim is proved, we are done: each occurrence of Y^+ in B_n is linked by an *ax*-edge to an occurrence of Y^- in $C^\perp$. All these occurrences of Y^- are pairwise distinct (recall Remark 8): there is no sub-formula $?D$ of $C^\perp$ containing such an occurrence, so they cannot be contracted; and there is no &-rule that can share the identified *ax*-rules since we are in a 0-slice, and

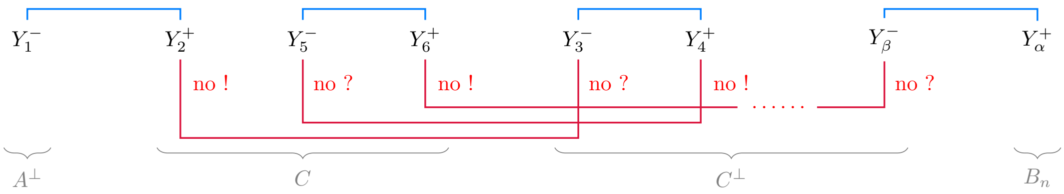


Figure 2 Alternating path p in the GOI projection graph $\mathcal{G}_\mu$ in the proof of Proposition 13

these ax-rules cannot be above a !-rule for there is no ?-connective nor !-connective in B_n. Therefore, we have found $n+2$ occurrences of Y^- in $C^\perp$, as wished.

Exploiting the alternating path. We prove our claim by showing a stronger result:

- for all occurrences of Y^- in C (respectively in $C^\perp$) belonging to p, there is no ? between the root of C (respectively of $C^\perp$) and these occurrences;
- for all occurrences of Y^+ in C (respectively in $C^\perp$) belonging to p, there is no ! between the root of C (respectively of $C^\perp$) and these occurrences.

We proceed by following the edges of p starting from its endpoint Y_1^- in $A^\perp$.

Exploiting the alternating path: base case. Consider the first edge Y_1^-–Y_2^+ of p, with Y_2^+ necessarily in C. Using Lemma 12, without any loss of generality, there is no !-connective between the root of C and Y_2^+. Otherwise, since there is an ax-rule on $\vdash Y_1^-, Y_2^+$, there would be a !-rule (associated to this !-connective) with in its conclusion sequent a ?-sub-formula of $A^\perp$ containing Y_1^-, meaning with $?(X^+ \otimes (X^- \parr Y^-))$: contradiction.

Exploiting the alternating path: inductive case. See Figure 2 for an illustration of our present reasoning. Assume the results holds for a prefix of p, and let us consider its last edge.

1. Suppose this prefix ends with an ax-edge Y_k^-–Y_{k+1}^+ with Y_{k+1}^+ in C. We know there is no ! between the root of C and Y_{k+1}^+. As p is alternating, the subsequent edge is a cut-edge Y_{k+1}^+–Y_{k+2}^- with Y_{k+2}^- in $C^\perp$. Since Y_{k+1}^+ and Y_{k+2}^- are dual occurrences by definition of a cut-edge, there is no ? between the root of $C^\perp$ and Y_{k+2}^-.
2. Suppose this prefix ends with a cut-edge Y_k^+–Y_{k+1}^- with Y_{k+1}^- in $C^\perp$. We know there is no ? between the root of $C^\perp$ and Y_{k+1}^-. As p is alternating, the subsequent edge is an ax-edge Y_{k+1}^-–Y_{k+2}^+ with Y_{k+2}^+ either in B_n or in $C^\perp$. In the first case, we are done: we have $Y_{k+2}^+ = Y_\alpha^+$ since there is no cut-edge with endpoint Y_{k+2}^+ to continue our path, and p is supposed to end in Y_α^+. In the second case, since there is an ax-rule $\dfrac{}{\vdash Y_{k+1}^-, Y_{k+2}^+}(ax)$ by definition of an ax-edge, it follows there is no ! between the root of $C^\perp$ and Y_{k+2}^+ – because the context condition of a corresponding !-rule would prevent the presence of Y_{k+1}^- in all sequents above it.
3. Suppose this prefix ends with an ax-edge Y_k^-–Y_{k+1}^+ with Y_{k+1}^+ in $C^\perp$. This case is identical to case 1 up to switching C and $C^\perp$.
4. Suppose this prefix ends with a cut-edge Y_k^+–Y_{k+1}^- with Y_{k+1}^- in C. This case is identical to case 2, with no need to consider Y_{k+2}^+ in $A^\perp$ since $A^\perp$ has no Y^+.

This proves our claim, and concludes the proof. ◀

▶ Remark 14. By going in the other direction of the path p, we can similarly show that:

- for all occurrences of Y^- in C (respectively in $C^\perp$) belonging to p, there is no ! between the root of C (respectively of $C^\perp$) and these occurrences;

- for all occurrences of Y^+ in C (respectively in $C^\perp$) belonging to p, there is no ? between the root of C (respectively of $C^\perp$) and these occurrences.

This means the "useful" part of C as identified by p is exponential-free.

Now that we have all needed properties, we easily obtain our main result.

▶ **Theorem 15.** *Linear logic does not have the uniform interpolation property.*

Proof. The formula A cannot have a uniform interpolant C with respect to X, as such a C would have an unbounded number of occurrences of Y^+ by Lemma 11 and Proposition 13. ◀

5 Adaptation to the Intuitionistic and Affine variants

Our counter-example to the uniform interpolation property in linear logic is also a counter-example to it in *intuitionistic* linear logic [14]. This can be directly obtained through a result of Harold Schellinx [21]. Formulas of propositional intuitionistic linear logic are given by:

$$A, B ::= X^+ \mid A \multimap B \mid A \otimes B \mid 1 \mid A \,\&\, B \mid A \oplus B \mid \top \mid 0 \mid !A$$

▶ **Proposition 16** ([21, Proposition 3.8])**.** *Consider a formula D of intuitionist linear logic which does not contain 0 or does not contain $\multimap$. Then $\vdash D$ is provable in linear logic if and only if it is provable in intuitionistic linear logic.*

▶ **Theorem 17.** *Intuitionistic linear logic does not have the uniform interpolation property.*

Proof. The formula A from Equation (1) has no uniform interpolant with respect to X. Indeed, $\vdash A \multimap B_n$ is provable intuitionistically for all $n \in \mathbb{N}$ using Lemma 11 and Proposition 16 – for 0 is not in $A \multimap B_n$. This yields an intuitionistic derivation of $A \vdash B_n$:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(ax) \qquad \dfrac{}{B_n \vdash B_n}\,(ax)}{A, A \multimap B_n \vdash B_n}\,(\multimap) \qquad \vdash A \multimap B_n}{A \vdash B_n}\,(cut)$$

Hence, we conclude as in the classical case: an interpolant for A would have an unbounded number of occurrences of Y^+ by Proposition 13 and conservativity – *i.e.* a sequent provable in intuitionistic linear logic is *a fortiori* provable in linear logic. ◀

Our counter-example is also one to the uniform interpolation property in *affine* linear logic, both classical and intuitionistic. Affine logic is defined as linear logic where the $?w$-rule is replaced by the more general $\dfrac{\vdash \Gamma}{\vdash A, \Gamma}\,(w)$. It is equipped with a weakly normalising cut-elimination procedure – see Appendix B. We need here one more intermediate result.

▶ **Lemma 18.** *In affine logic, there is no derivation of $\Gamma \vdash \bot$ where Γ is a sequent whose formulas are included in $\{X, !(X \multimap (X \otimes Y)), X \multimap (X \otimes Y), X \otimes Y, Y, X \multimap Y\}$, with a same formula appearing possibly several times in Γ.*

Proof. By contradiction, consider a smallest cut-free derivation of $\Gamma \vdash \bot$ with Γ a sequent of the aforementioned shape. Its last rule is either a $\otimes$- $\parr$- $?d$- $?c$- or w-rule. In each case, one of the premises of this rule is a sequent of the aforementioned shape, which contradicts the minimality hypothesis. ◀

▶ **Theorem 19.** *Affine logic and intuitionistic affine logic do not have the uniform interpolation property.*

Proof. We proceed almost exactly as for linear logic, the sole difference being in the beginning of the proof of Proposition 13. There, that in the normalised derivation τ of $\vdash A^{\perp}, B_n$ (or $A \vdash B_n$) there is an *ax*-rule on each Y^+ of B_n is no longer immediate: we invoke Lemma 18, which implies no sub-formula of B_n can be weakened in τ – this lemma holding in affine logic, so *a fortiori* in intuitionistic affine logic by conservativity. ◀

6 Conclusion

We presented a formula of unit-free multiplicative-exponential linear logic that has no uniform interpolant in linear logic, both classical and intuitionistic, as well as in classical and intuitionistic affine logic. This contrasts sharply with the situation of classical logic and intuitionistic logic and multiplicative-additive linear logic, that all have the uniform interpolation property. Our counter-example sheds some light on when the contraction rule prevents uniform interpolation: it may do so when the formula A to interpolate contains both a non-contractible sub-formula (here X^+) and a contractible sub-formula (here $!(X^+ \multimap (X^+ \otimes Y^+))$), which is impossible in the three aforementioned logics but possible in linear logic and affine logic! Furthermore, please observe that our demonstration also holds in a variant of linear logic with no $?w$-rule and/or with the $?c$-rule replaced by the weaker selection rule $\dfrac{\vdash A, ?A, \Gamma}{\vdash ?A, \Gamma}$. This makes quite clear that what prevents uniform interpolation in linear logic is a contraction rule available only on some formulas (along with the constraint on the context in a !-rule).

The formulas A and B_n we exhibited in Equation (1) are not overly complex, but proving no formula C can be a uniform interpolant was quite tedious as it depends heavily on the necessary structure of C. Geometry of interaction and slices were very useful tools to this end – and we do not do how to prove Proposition 13 without GOI projection graphs.

A natural question following the failure of the uniform interpolation property is the subsequent decision problem: given a formula A, does it have a uniform interpolant? This is an open problem for formulas of (multiplicative-exponential) (intuitionistic) linear logic. Considering the uniform interpolation property in linear logic and its fragments, since it trivially holds in the presence of quantifiers (setting $C = \exists X A$ as an interpolant for A with respect to X), and it has been proved in multiplicative-additive linear logic [1] and now been disproved for some formula of multiplicative-exponential linear logic (even allowing the interpolant to use connectives from propositional linear logic), the principal fragment where the question stays open is the (unusual) additive-exponential linear logic. Since in this logic the contraction rule is of limited interest due to the lack of multiplicative connectives, we conjecture the method of Pitts [18] could be adapted to prove uniform interpolation in this fragment, as well as in its intuitionistic and affine variants.

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A Cut-Elimination in Linear Logic

Cut-elimination in linear logic is defined on Tables 1 and 2. It is weakly normalising [11].

ax	$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \overset{\pi}{\vdash A,\Gamma}}{\vdash A,\Gamma}\,(cut)$	$\longrightarrow$	$\overset{\pi}{\vdash A,\Gamma}$
$\parr - \otimes - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B^\perp,\Gamma}}{\vdash A^\perp \parr B^\perp,\Gamma}\,(\parr) \qquad \dfrac{\overset{\phi}{\vdash A,\Delta} \quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B,\Delta,\Sigma}\,(\otimes)}{\vdash \Gamma,\Delta,\Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B^\perp,\Gamma}\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A^\perp,\Gamma,\Sigma}\,(cut) \qquad \overset{\phi}{\vdash A,\Delta}}{\vdash\Gamma,\Delta,\Sigma}\,(cut)$
$\parr - \otimes - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B^\perp,\Gamma}}{\vdash A^\perp \parr B^\perp,\Gamma}\,(\parr) \qquad \dfrac{\overset{\phi}{\vdash A,\Delta} \quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B,\Delta,\Sigma}\,(\otimes)}{\vdash \Gamma,\Delta,\Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B^\perp,\Gamma}\quad \overset{\phi}{\vdash A,\Delta}}{\vdash B^\perp,\Gamma,\Sigma}\,(cut) \qquad \overset{\tau}{\vdash B,\Sigma}}{\vdash\Gamma,\Delta,\Sigma}\,(cut)$
$\bot - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash\Gamma}}{\vdash\bot,\Gamma}\,(\bot)\qquad \dfrac{}{\vdash 1}\,(1)}{\vdash\Gamma}\,(cut)$	$\longrightarrow$	$\overset{\pi}{\vdash\Gamma}$
$\& - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}\quad\overset{\phi}{\vdash B^\perp,\Gamma}}{\vdash A^\perp \& B^\perp,\Gamma}\,(\&) \qquad \dfrac{\overset{\tau}{\vdash A,\Delta}}{\vdash A\oplus B,\Delta}\,(\oplus_1)}{\vdash\Gamma,\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\phi}{\vdash A^\perp,\Gamma}\quad\overset{\tau}{\vdash A,\Delta}}{\vdash\Gamma,\Delta}\,(cut)$
$\& - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}\quad\overset{\phi}{\vdash B^\perp,\Gamma}}{\vdash A^\perp \& B^\perp,\Gamma}\,(\&) \qquad \dfrac{\overset{\tau}{\vdash B,\Delta}}{\vdash A\oplus B,\Delta}\,(\oplus_2)}{\vdash\Gamma,\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash B^\perp,\Gamma}\quad\overset{\tau}{\vdash B,\Delta}}{\vdash\Gamma,\Delta}\,(cut)$
$?d - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}}{\vdash ?A^\perp,\Gamma}\,(?d)\qquad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}\quad\overset{\phi}{\vdash A,?\Delta}}{\vdash\Gamma,?\Delta}\,(cut)$
$?c - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp,?A^\perp,\Gamma}}{\vdash ?A^\perp,\Gamma}\,(?c)\qquad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp,?A^\perp,\Gamma}\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash ?A^\perp,\Gamma,?\Delta}\,(cut)\qquad \dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta,?\Delta}\,(cut)}{\vdash\Gamma,?\Delta}\,(?c)$
$?w - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash\Gamma}}{\vdash ?A^\perp,\Gamma}\,(?w)\qquad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash\Gamma}}{\vdash\Gamma,?\Delta}\,(?w)$

Table 1 Cut-elimination in Linear Logic – Key cases

$cut - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B^\perp, \Gamma, \Delta}\,(cut) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma}\,(cut) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$
$\parr - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, C, \Gamma}}{\vdash A^\perp, B \parr C, \Gamma}\,(\parr) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B \parr C, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, C, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, C, \Gamma, \Delta}\,(cut)}{\vdash B \parr C, \Gamma, \Delta}\,(\parr)$
$\otimes - cut - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\phi}{\vdash C, \Delta}}{\vdash A^\perp, B \otimes C, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B, \Gamma, \Sigma}\,(cut) \quad \overset{\phi}{\vdash C, \Delta}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\otimes)$
$\otimes - cut - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \overset{\phi}{\vdash A^\perp, C, \Delta}}{\vdash A^\perp, B \otimes C, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash B, \Gamma} \quad \dfrac{\overset{\phi}{\vdash A^\perp, C, \Delta} \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash C, \Delta, \Sigma}\,(cut)}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\otimes)$
$\perp - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma}}{\vdash A^\perp, \perp, \Gamma}\,(\perp) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \perp, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \perp, \Gamma, \Delta}\,(\perp)$
$\& - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\phi}{\vdash A^\perp, C, \Gamma}}{\vdash A^\perp, B \,\&\, C, \Gamma}\,(\&) \quad \overset{\tau}{\vdash A, \Delta}}{\vdash B \,\&\, C, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut) \quad \dfrac{\overset{\phi}{\vdash A^\perp, C, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash C, \Gamma, \Delta}\,(cut)}{\vdash B \,\&\, C, \Gamma, \Delta}\,(\&)$
$\oplus_1 - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma}}{\vdash A^\perp, B \oplus C, \Gamma}\,(\oplus_1) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B \oplus C, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash B \oplus C, \Gamma, \Delta}\,(\oplus_1)$
$\oplus_2 - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, C, \Gamma}}{\vdash A^\perp, B \oplus C, \Gamma}\,(\oplus_2) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B \oplus C, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, C, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash C, \Gamma, \Delta}\,(cut)}{\vdash B \oplus C, \Gamma, \Delta}\,(\oplus_2)$
$\top - cut$	$\dfrac{\dfrac{}{\vdash A^\perp, \top, \Gamma}\,(\top) \quad \overset{\pi}{\vdash A, \Delta}}{\vdash \top, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{}{\vdash \top, \Gamma, \Delta}\,(\top)$
$?d - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma}}{\vdash A^\perp, ?B, \Gamma}\,(?d) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?d)$
$?c - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, ?B, ?B, \Gamma}}{\vdash A^\perp, ?B, \Gamma}\,(?c) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, ?B, ?B, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, ?B, \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?c)$
$?w - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma}}{\vdash A^\perp, ?B, \Gamma}\,(?w) \quad \overset{\phi}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma} \quad \overset{\phi}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?w)$
$! - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp, B, ?\Gamma}}{\vdash ?A^\perp, !B, ?\Gamma}\,(!) \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash !B, ?\Gamma, ?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp, B, ?\Gamma} \quad \dfrac{\overset{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash B, ?\Gamma, ?\Delta}\,(cut)}{\vdash !B, ?\Gamma, ?\Delta}\,(!)$

Table 2 Cut-elimination in Linear Logic – Commutative cases

B Cut-Elimination in Affine Logic

Cut-elimination in affine logic is defined as in linear logic, except that the $?w-!$ key case and the $?w - cut$ commutative case are replaced by the two cases on Table 3. That cut-elimination is weakly normalising in affine logic can be proved similarly to weak normalisation of cut-elimination in linear logic – it is *e.g.* a simple adaptation of the proof in [16].

w	$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash A, \Gamma}\,(w) \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \Gamma, \Delta}\,(w)$ (double line)
$w - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma}}{\vdash A, B, \Gamma}\,(w) \qquad \overset{\phi}{\vdash A^{\perp}, \Delta}}{\vdash \Gamma, B, \Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\phi}{\vdash A^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \Gamma, B, \Delta}\,(w)$

Table 3 Cut-elimination in Affine Logic – Different cases with respect to Linear Logic