

Cut-Cut Commutations Are Not Completely Superfluous

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Abstract

In the definition of a cut-elimination procedure, there often is a *cut – cut* commutation allowing to swap two *cut*-rules. This allows for instance to fully eliminate a chosen *cut*-rule without reducing other *cut*-rules in a proof. One may wonder whether such *cut – cut* steps are superfluous. We prove it is not in linear logic: some cut-free forms of a proof can only be reached using a *cut – cut* commutation.

1 Introduction

We consider the sequent calculus of linear logic [Gir87]. As many logics, it has a *cut*-rule and a rewriting system called *cut-elimination*, which details how to reach a cut-free (or normal) proof starting from a proof with possibly many *cut*-rules. Cut-elimination has been studied extensively in this system and more generally in linear logic (but mainly for its proof-net syntax): mostly its normalization [Acc13; DG99; LM08; Tor03; PT10] but also its confluence [CP05; Di24]. In particular, it is well-known one can reach a normal form without using any *cut – cut* commutations. It can also be proved that equality up to cut-elimination is the same as equality up to cut-elimination without the *cut – cut* commutation. One may wonder whether such *cut – cut* steps are completely superfluous: can the same normal forms be reached with and without *cut – cut* commutations? We answer this question negatively, exhibiting a simple counter-example. This counter-example is even one for the simpler multiplicative-exponential fragment of linear logic, that we will consider here for simplicity's sake.

2 Definitions

We consider the sequent calculus of multiplicative-exponential linear logic [Gir87]. Formulas are given by the following grammar, with X an atom in a given countable set:

$$A, B ::= X^+ \mid X^- \mid A \wp B \mid A \otimes B \mid \perp \mid 1 \mid ?A \mid !A$$

We define on formulas a function $(\cdot)^\perp$ called **orthogonality**, also named negation or duality, through the following inductive definition:

$$\begin{array}{c}
\frac{}{\vdash A^\perp, A} \text{ (ax)} \quad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)} \\
\frac{\vdash A, B, \Gamma}{\vdash A \wp B, \Gamma} \text{ (?)} \quad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \text{ (?)} \quad \frac{\vdash \Gamma}{\vdash \perp, \Gamma} \text{ (?)} \quad \frac{}{\vdash 1} \text{ (?)} \\
\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma} \text{ (?d)} \quad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma} \text{ (?c)} \quad \frac{\vdash \Gamma}{\vdash ?A, \Gamma} \text{ (?w)} \quad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma} \text{ (!)}
\end{array}$$

Figure 1: Rules of Linear Logic

$$\begin{array}{ll}
(X^+)^\perp = X^- & (X^-)^\perp = X^+ \\
(A \wp B)^\perp = B^\perp \otimes A^\perp & (A \otimes B)^\perp = B^\perp \wp A^\perp \\
\perp^\perp = 1 & 1^\perp = \perp \\
(?A)^\perp = !A^\perp & (!A)^\perp = ?A^\perp
\end{array}$$

Sequents are sets of (occurrences of) formulas written as $\vdash A_1, \dots, A_n$. Rules of linear logic are given on Fig. 1, where A and B stand for arbitrary formulas, Γ and Δ for sets of (occurrences of) formulas. By $? \Gamma$ we mean that each formula of Γ is a $?$ -formula.

As in many systems with a *cut*-rule, the *cut*-rule is *admissible* in this logic: the same sequents can be proved with and without the *cut*-rule. *Cut-elimination* is the procedure turning a proof into a *cut*-free one. As we will not use *ax*- \otimes - \wp - $?_c$ - and $?_w$ -rules, we give this rewriting system without those. For a full description, see [Di24, Chapter 1].

Definition 1. Cut-elimination is the rewriting system whose rules are described on Table 1, up to commuting the two branches of any *cut*-rule.

The “up to commutation” means we consider a version of each case with the left and right premises of any *cut*-rule swapped – *e.g.* the *cut* – *cut* case gives 4 rewriting rules.

3 Cut-cut commutations allow to reach more normal forms

We show that removing the *cut* – *cut* step in multiplicative-exponential linear logic (and thus in linear logic) narrows the set of normal forms.

Lemma 2. *There exist a proof π and a cut-free proof ϕ such that π reduces by cut-elimination to ϕ using a cut – cut elimination step (and other cut elimination steps), but π does not reduce to ϕ without using a cut – cut elimination step.*

$\perp - 1$	$\frac{\frac{\pi}{\frac{\frac{\perp}{\vdash \perp, \Gamma} (\perp) \quad \frac{\overline{\vdash 1}}{\vdash 1} (1)}{\vdash \Gamma} (cut)}}{\vdash \Gamma}$	$\xrightarrow{\beta}$	$\frac{\pi}{\vdash \Gamma}$
$?d - !$	$\frac{\frac{\frac{\pi}{\frac{\frac{\perp}{\vdash A^\perp, \Gamma} (?d) \quad \frac{\phi}{\vdash A, ?\Delta} (!)}{\vdash ?A^\perp, \Gamma} (cut)}}{\vdash \Gamma, ?\Delta}$	$\xrightarrow{\beta}$	$\frac{\frac{\pi}{\vdash A^\perp, \Gamma} \quad \frac{\phi}{\vdash A, ?\Delta}}{\vdash \Gamma, ?\Delta} (cut)$
$\perp - cut$	$\frac{\frac{\frac{\pi}{\frac{\frac{\perp}{\vdash A^\perp, \Gamma} (\perp) \quad \frac{\phi}{\vdash A, \Delta} (cut)}}{\vdash \perp, \Gamma, \Delta}$	$\xrightarrow{\beta}$	$\frac{\frac{\pi}{\vdash A^\perp, \Gamma} \quad \frac{\phi}{\vdash A, \Delta}}{\frac{\vdash \Gamma, \Delta}{\vdash \perp, \Gamma, \Delta} (\perp)} (cut)$
$?d - cut$	$\frac{\frac{\frac{\pi}{\frac{\frac{\perp}{\vdash A^\perp, B, \Gamma} (?d) \quad \frac{\phi}{\vdash A, \Delta} (cut)}}{\vdash ?B, \Gamma, \Delta}$	$\xrightarrow{\beta}$	$\frac{\frac{\pi}{\vdash A^\perp, B, \Gamma} \quad \frac{\phi}{\vdash A, \Delta}}{\frac{\vdash B, \Gamma, \Delta}{\vdash ?B, \Gamma, \Delta} (?d)} (cut)$
$! - cut$	$\frac{\frac{\frac{\pi}{\frac{\frac{\perp}{\vdash ?A^\perp, B, ?\Gamma} (!) \quad \frac{\phi}{\vdash A, ?\Delta} (!)}{\vdash ?A^\perp, !B, ?\Gamma} (cut)}}{\vdash !B, ?\Gamma, ?\Delta}$	$\xrightarrow{\beta}$	$\frac{\frac{\frac{\pi}{\vdash ?A^\perp, B, ?\Gamma} \quad \frac{\frac{\phi}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta} (!)}{\vdash B, ?\Gamma, ?\Delta} (!)}{\vdash !B, ?\Gamma, ?\Delta} (cut)$
$cut - cut$	$\frac{\frac{\frac{\frac{\pi}{\frac{\frac{\perp}{\vdash A^\perp, B^\perp, \Gamma} \quad \frac{\phi}{\vdash A, \Delta} (cut)}}{\vdash B^\perp, \Gamma, \Delta} \quad \frac{\tau}{\vdash B, \Sigma} (cut)}}{\vdash \Gamma, \Delta, \Sigma}$	$\xrightarrow{\beta}$	$\frac{\frac{\frac{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \frac{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma} (cut) \quad \frac{\phi}{\vdash A, \Delta} (cut)}{\vdash \Gamma, \Delta, \Sigma}$

Table 1: Cut-elimination \perp , 1 , $?_d$ and $!$ cases

Proof. Set π the following proof:

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\overline{\vdash 1}}{\vdash 1} (1)}{\vdash 1, \perp} (\perp)}{\vdash 1, ?\perp} (?d)}{\vdash ?1, ?\perp} (?d)}{\vdash \perp, ?1, ?\perp} (\perp)}{\frac{\frac{\overline{\vdash 1}}{\vdash 1} (1) \quad \frac{\frac{\frac{\frac{\frac{\frac{\overline{\vdash 1}}{\vdash 1} (!)}{\vdash !1} (!)}{\vdash !1, \perp} (\perp)}{\vdash !\perp, ?1, ?\perp} (!)}{\vdash !\perp, ?1, \perp} (cut)}}{\vdash ?1, \perp} (cut)}$$

(This proof with an interactive cut-elimination procedure is available here using the interactive linear logic prover `C1ick` \wp `c` \otimes `LLec` \perp .)

Without using *cut – cut* steps, cut-elimination on π leads to a unique normal form:

$$\frac{\frac{\overline{\quad} (1)}{\vdash 1} (?_d)}{\vdash ?1} (\perp)$$

For our purpose, it is enough to prove any normal form of π reached without a *cut – cut* step has a \perp -rule at its root. This is the case: one first has to apply a $\perp – cut$ step on the upper *cut*-rule of π ; then, whatever happens on the upper *cut*-rule, the first step involving the bottom *cut*-rule must be a $\perp – cut$ step, resulting in a \perp -rule at the root of the proof. Meanwhile, consider the result of applying a *cut – cut* step first:

$$\frac{\frac{\frac{\overline{\quad} (1)}{\vdash 1} (\perp)}{\vdash 1, \perp} (?_d)}{\vdash 1, ?\perp} (?_d)}{\vdash ?1, ?\perp} (\perp)}{\frac{\frac{\overline{\quad} (1)}{\vdash 1} (?_d)}{\vdash ?1} (\perp)}{\vdash ?1, ?\perp} (!)}{\vdash ?1, ?\perp} (cut)}{\frac{\frac{\overline{\quad} (1)}{\vdash 1} (!)}{\vdash !1} (\perp)}{\vdash !1, \perp} (!)}{\vdash ?1, \perp} (cut)}$$

Here, one can apply a $?_d – !$ step on the upper *cut*-rule, then a $\perp – 1$ step still on this *cut*-rule, followed by a $?_d – cut$ step on the (bottom) *cut*-rule, leading to a proof with a $?_d$ -rule at its root. In particular, one can reach the following normal form ϕ :

$$\frac{\frac{\overline{\quad} (1)}{\vdash 1} (\perp)}{\vdash 1, \perp} (?_d)}{\vdash ?1, \perp}$$

□

4 Conclusion

Perhaps surprisingly, *cut – cut* commutative steps of cut-elimination are not totally useless: they allow to reach more normal forms in multiplicative-exponential linear logic, and so in linear logic. Nonetheless, our example seems hard to adapt without a contextual rule such as $!$ – which needs the sequent it is applied on to be of the shape $!A, ?\Gamma$. Hence, we conjecture that in the multiplicative-additive fragment of linear logic the same normal forms can be reached with and without *cut – cut* commutative steps.

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