Proof Equivalence in Linear Logic is Undecidable

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August 19, 2025

Abstract

Proofs of linear logic are often considered up to rule commutations. We prove that knowing whether two given proofs are equal up to rule commutations, or not, is undecidable.

1 Introduction

We consider the sequent calculus of linear logic [Gir87]. Usually, one does not want to consider proofs of linear logic up to syntactic equality but up to an equivalence relation called **rule commutation**. These commutations associate proofs which differ only by the order in which their rules are applied. In other words, this relation appears when one can apply two successive rules, say r and s, and both applying r then s or s then r would work. That is why rule commutations can be viewed as "bureaucracy" [Gir01], as both choices are adequate, and in fact are not really a choice as we wish to consider both proofs the same. More formally, equating proofs up to rule commutations is the same as equating proofs up to cut-elimination – at least in multiplicative-additive linear logic [CP05; Di24] – which is often done semantically. But due to the definition of a proof as a tree of rules, we must choose an arbitrary order and apply one of the two rules first, for it is impossible to apply two rules at once in sequent calculus. This is why proof-nets [Gir96] are an interesting syntax: they allow to apply two rules at the same time, and equate proofs exactly up to rule commutations [HG16].

We consider in this note the **proof equivalence problem**: given two proofs, are they equal up to rule commutations or not? This problem is already known to be PSPACE-complete in the multiplicative fragment of linear logic [HH16]. We prove undecidability of the proof equivalence problem through a reduction from the **provability problem**: given a sequent, is it provable or not? The complexity of the provability problem is well-known for many fragments of linear logic [Lin95; Chu21; HH15], and in particular is undecidable for propositional linear logic.

$$
\frac{}{\vdash A^\perp, A}\,(ax) \qquad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\,(cut)
$$

$$
\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,(\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,(\bot) \qquad \frac{}{\vdash 1}\,(1)
$$

$$
\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\,(\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\,(\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\,(\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\,(\top)
$$

$$
\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,(?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,(?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,(?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,(!)
$$

Figure 1: Rules of Linear Logic

2 Definitions

We define here the logical system under consideration: the sequent calculus of propositional linear logic [Gir87]. Formulas are given by the following grammar, where X is an atom in a given countable set:

$$
\begin{aligned}
A, B ::= {} & \mid X^+ \mid X^- && \text{(atom)} \\
& \mid A \parr B \mid A \otimes B \mid \bot \mid 1 && \text{(multiplicative)} \\
& \mid A \,\&\, B \mid A \oplus B \mid \top \mid 0 && \text{(additive)} \\
& \mid ?A \mid !A && \text{(exponential)}
\end{aligned}
$$

We define on formulas a function $(\cdot)^\perp$ called **orthogonality**, also named negation or duality, through the following inductive definition:

$$
\begin{aligned}
(X^+)^\perp &= X^- & (X^-)^\perp &= X^+ \\
(A \parr B)^\perp &= B^\perp \otimes A^\perp & (A \otimes B)^\perp &= B^\perp \parr A^\perp \\
\bot^\perp &= 1 & 1^\perp &= \bot \\
(A \,\&\, B)^\perp &= B^\perp \oplus A^\perp & (A \oplus B)^\perp &= B^\perp \,\&\, A^\perp \\
\top^\perp &= 0 & 0^\perp &= \top \\
(?A)^\perp &= !A^\perp & (!A)^\perp &= ?A^\perp
\end{aligned}
$$

Sequents are multisets of formulas written in the form $\vdash A_1, \ldots, A_n$. Rules of linear logic are given on Figure 1, where A and B stand for arbitrary formulas, Γ and Δ for multisets of formulas. The notation $?\Gamma$ means that each formula of Γ is a $?$-formula.

Rule commutation is the equivalence relation denoted $\overset{r}{\longmapsto\!\!\dashv}$ which is the equivalence closure of the rules obtained the following way: consider a *cut*-rule c and all couples (r, s) of non *cut*-rules that can be the premises of c, such that two commutative cut-elimination cases can be applied on c; comparing the results of applying the commutative step with

$\otimes - \oplus_i - 1$	$\cfrac{\cfrac{\overset{\pi}{\vdash A_1, B_i, \Gamma} \quad \overset{\phi}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_i, \Gamma, \Delta}\,(\otimes)}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,(\oplus_i)$	$\overset{r}{\vdash\!\dashv}$	$\cfrac{\cfrac{\overset{\pi}{\vdash A_1, B_i, \Gamma}}{\vdash A_1, B_1 \oplus B_2, \Gamma}\,(\oplus_i) \quad \overset{\phi}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,(\otimes)$
$\otimes - \oplus_i - 2$	$\cfrac{\cfrac{\overset{\pi}{\vdash A_1, \Gamma} \quad \overset{\phi}{\vdash A_2, B_i, \Delta}}{\vdash A_1 \otimes A_2, B_i, \Gamma, \Delta}\,(\otimes)}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,(\oplus_i)$	$\overset{r}{\vdash\!\dashv}$	$\cfrac{\overset{\pi}{\vdash A_1, \Gamma} \quad \cfrac{\overset{\phi}{\vdash A_2, B_i, \Delta}}{\vdash A_2, B_1 \oplus B_2, \Delta}\,(\oplus_i)}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,(\otimes)$
$\otimes - \top - 1$	$\cfrac{}{\vdash A_1 \otimes A_2, \top, \Gamma, \Delta}\,(\top)$	$\overset{r}{\vdash\!\dashv}$	$\cfrac{\cfrac{}{\vdash A_1, \top, \Gamma}\,(\top) \quad \overset{\pi}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, \top, \Gamma, \Delta}\,(\otimes)$
$\otimes - \top - 2$	$\cfrac{}{\vdash A_1 \otimes A_2, \top, \Gamma, \Delta}\,(\top)$	$\overset{r}{\vdash\!\dashv}$	$\cfrac{\overset{\pi}{\vdash A_1, \Gamma} \quad \cfrac{}{\vdash A_2, \top, \Delta}\,(\top)}{\vdash A_1 \otimes A_2, \top, \Gamma, \Delta}\,(\otimes)$
$\oplus_i - \top$	$\cfrac{}{\vdash A_1 \oplus A_2, \top, \Gamma}\,(\top)$	$\overset{r}{\vdash\!\dashv}$	$\cfrac{\cfrac{}{\vdash A_i, \top, \Gamma}\,(\top)}{\vdash A_1 \oplus A_2, \top, \Gamma}\,(\oplus_i)$
$\top - \top$	$\cfrac{}{\vdash \top^1, \top^2, \Gamma}\,(\top^1)$	$\overset{r}{\vdash\!\dashv}$	$\cfrac{}{\vdash \top^1, \top^2, \Gamma}\,(\top^2)$

Table 1: Some cases of rule commutations

r then with s, against applying the one with s then with r, yields the rule commutation between r and s. In particular, rule commutations contain the cases on Table 1. These few rules are enough for our proof, in addition to the following fact.

Fact 1. *For any formula A such that $!A$ is not provable, no rule commutation can be applied on* $\cfrac{}{\vdash !A, \top}\,(\top)$.

We use two notations on Table 1. First, by a $\oplus_i$-rule we mean either a $\oplus_1$- or a $\oplus_2$-rule – *i.e.* $i \in \{1; 2\}$. Second, in the $\top - \top$ commutation we need to distinguish the two occurrences of $\top$ to know which $\top$-formula is associated to the $\top$-rule. We do so by adding exponents to $\top$-formulas and $\top$-rules, here and when needed in the rest of this note. For a full description of rule commutations with all its cases, see [Di24, Chapter 1]. **Proof equivalence** is the problem of deciding whether two given proofs are equivalent, *i.e.* related by $\overset{r}{\vdash\!\dashv}$.

3 Undecidability of proof equivalence

We prove undecidability by reducing the provability problem to the proof equivalence problem (Lemma 3). Our proof is simple and only uses an elementary fact.

Fact 2. *A formula A is provable if and only if $!A$ is provable.*

Proof. If π is a proof of A then we get a proof of $!A$ using a !-rule: $\dfrac{\overset{\pi}{\vdash A}}{\vdash\ !A}\,(!)$. Conversely, if π is a proof of $!A$ then we build a proof of A: $\dfrac{\overset{\pi}{\vdash\ !A} \qquad \dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax)}{\vdash\ ?A^{\perp}, A}\,(?_d)}{\vdash A}\,(cut)$. ☐

Lemma 3. *For A any formula, the following holds:*

$$\left(\dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\,(\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\,(\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\,(\oplus_2)\right) \iff A \text{ is provable}$$

Proof. If A is provable, here is a sequence of rule commutations between these two proofs:

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top^A, \top}\,(\top)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{\overset{\pi}{\vdash\ !A} \qquad \dfrac{}{\vdash \top^A, \top}\,(\top)}{\vdash\ !A \otimes \top^A, \top}\,(\otimes)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_i) \quad \text{(Fact 2 giving a proof } \pi \text{ of } !A)$$

$$\overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{\overset{\pi}{\vdash\ !A} \qquad \dfrac{}{\vdash \top^A, \top}\,(\top^A)}{\vdash\ !A \otimes \top^A, \top}\,(\otimes)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_i)$$

$$\overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\overset{\pi}{\vdash\ !A} \qquad \dfrac{\dfrac{}{\vdash \top^A, \top}\,(\top^A)}{\vdash \top^A, \top \oplus \top}\,(\oplus_i)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\otimes)$$

$$\overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\overset{\pi}{\vdash\ !A} \qquad \dfrac{}{\vdash \top^A, \top \oplus \top}\,(\top^A)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\otimes)$$

Thence:

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top^A, \top}\,(\top)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_1) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\overset{\pi}{\vdash\ !A} \qquad \dfrac{}{\vdash \top^A, \top \oplus \top}\,(\top^A)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\otimes) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{}{\vdash\ !A \otimes \top^A, \top}\,(\top)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_2)$$

If A is not provable, then we compute the equivalence classes of both proofs:

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top^A, \top}\,(\top)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\,(\top) \qquad \dfrac{}{\vdash \top^A}\,(\top^A)}{\vdash\ !A \otimes \top^A, \top}\,(\otimes)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\,(\top)}{\vdash\ !A, \top \oplus \top}\,(\oplus_i) \qquad \dfrac{}{\vdash \top^A}\,(\top^A)}{\vdash\ !A \otimes \top^A, \top \oplus \top}\,(\otimes)$$

using that there is no proof of $!A$ by Fact 2 and no further commutations by Fact 1. As these two classes are different, the two proofs are not related by rule commutations. ☐

Proposition 4. *Proof equivalence in propositional linear logic is undecidable.*

Proof. It is known that provability in this proof system is undecidable [Lin95]. By Lemma 3, decidability of proof equivalence would imply decidability of the provability of any formula – hence of any sequent by taking the $\parr$ of its formulas. $\square$

Remark 5. We can adapt our proof so as to not use the exponential ! up to some more technicalities. More precisely, we can prove that for any formula A and any atom X:

$$\left(\dfrac{\dfrac{}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top}\ (\top)}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top \oplus \top}\ (\oplus_1) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top}\ (\top)}{\vdash ((X^+ \parr X^-) \,\&\, A) \otimes \top, \top \oplus \top}\ (\oplus_2) \right)$$
$$\iff A \text{ is provable}$$

4 Conclusion

We proved that proof equivalence is undecidable in linear logic, by a simple reduction from provability. In particular, this completely prevents any canonical notion of proof-nets for full linear logic, as it implies either translating into a proof-net is undecidable, or equality of these proof-nets is undecidable. Adding the *mix*-rule(s) does not change this undecidability result, for our proof also extends to that setting (up to minor considerations due to the $\top - \mathit{mix}$ commutations). This contrasts the case of multiplicative linear logic [HH16] where adding the *mix*-rule turns the problem from PSPACE-complete to solvable by proof-nets (see [FR94] for the first definition of these proof-nets).

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