No Uniform Interpolation for (I)(ME)LL (without units)

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Abstract

Few results are known on uniform interpolation for Linear Logic, the main one being that Multiplicative-Additive Linear Logic has the uniform interpolation property, and so does its intuitionistic variant [ADO14]. We present a simple counter-example to this property for Linear Logic, with a formula of unit-free Multiplicative-Exponential Linear Logic that has no uniform interpolant, in both classical and intuitionistic Linear Logic.

1 Introduction

Uniform interpolation is a stronger version of Craig's interpolation where the interpolant depends only on one of the two formulas under consideration. More precisely, the uniform interpolation property states that, given a formula A and an atom X, there exists a formula C whose atoms are included in those of A except for X, and such that:

- $A \vdash C$ is provable;
- for all formulas B such that X does not appear in B and $A \vdash B$ is provable, the sequent $C \vdash B$ is also provable.

In particular, uniform interpolants are an interpretation of first-order quantifiers in the logic, as the formula C above behaves like $\exists X A$ (at least for provability purposes).

Not every logic has the uniform interpolant property, for instance the modal logic S4 does not [GZ95]. It is easy to check that classical logic has this property. A uniform interpolant for a formula A with respect to an atom X is simply $A[\top/X] \vee A[\bot/X]$, that is the disjunction of substituting the atom by true or false. A foundational work from Pitts [Pit92] proves that intuitionistic logic also has the uniform interpolation property. The algorithm to compute it is more complex than in the classical case, and the proof technique is based on a syntactic approach. The method of Pitts for intuitionistic logic was then adapted to many other logics. In particular, it was used in [ADO14] to prove uniform interpolation for the Lambek calculus, as well as for classical and intuitionistic multiplicative-additive linear logic (and also for the affine variants of these two). Nonetheless, this proof technique does not extend easily to full propositional linear logic. The main reason is the presence of the contraction rule, that has a premise sequent "bigger" than its conclusion sequent, and that necessitated a particular treatment in Pitts' paper [Pit92] so as to remove it from the calculus. Still, since classical logic and intuitionistic logic both have the uniform interpolation property, one could expect linear logic to also have it. We show it is not the case by exhibiting a counter-example. Actually, we present formulas A and $(B_n)_{n\in\mathbb{N}}$ such that:

1. an atom X appears in A but not in any of the B_n;
2. $\forall n \in \mathbb{N}$, $A \vdash B_n$ is provable;
3. for all formulas C such that X does not appear in C, for any number $n \in \mathbb{N}$, if $A \vdash C$ and $C \vdash B_n$ are both provable, then C is of size at least n.

The result immediately follows: A cannot have a uniform interpolant with respect to X, as such an interpolant would have an unbounded size. More precisely, the formulas A and $(B_n)_{n\in\mathbb{N}}$ belong to unit-free multiplicative-exponential linear logic, but there is no uniform interpolant C for A even in full propositional linear logic. Furthermore, this is a counter-example not only for linear logic, but also for its intuitionistic variant.

Outline. We first recall the syntax of linear logic in Section 2, along with a definition of uniform interpolation in this setting. Then, we define some useful tools in Section 3. At last, we present our counter-example in Section 4, and prove it cannot have a uniform interpolant using the tools from the preceding section.

2 Definitions

We use the standard presentation of linear logic as a unilateral sequent calculus [Gir87]. In particular, proofs are *derivations* whose conclusion sequent is of the shape $\vdash \Gamma$. The only exception to the use of this framework is the demonstration of Lemma 9, with derivations in the sequent calculus of *intuitionistic* linear logic, see *e.g.* [GL87].

Formulas are defined by this grammar, where X is an atom in a given countable set:

$$A, B ::= \underbrace{X^- \mid X^+}_{\text{atom}} \mid \underbrace{A \parr B \mid A \otimes B \mid \bot \mid 1}_{\text{multiplicative}} \mid \underbrace{A \,\&\, B \mid A \oplus B \mid \top \mid 0}_{\text{additive}} \mid \underbrace{?A \mid !A}_{\text{exponential}}$$

Orthogonality $(\cdot)^\perp$ (*a.k.a.* negation, duality) is an involution defined by:

$$\begin{array}{cccc}
(X^-)^\perp \overset{\text{def}}{=} X^+ & (A \parr B)^\perp \overset{\text{def}}{=} A^\perp \otimes B^\perp & (A \,\&\, B)^\perp \overset{\text{def}}{=} A^\perp \oplus B^\perp & (?A)^\perp \overset{\text{def}}{=} !A^\perp \\
(X^+)^\perp \overset{\text{def}}{=} X^- & (A \otimes B)^\perp \overset{\text{def}}{=} A^\perp \parr B^\perp & (A \oplus B)^\perp \overset{\text{def}}{=} A^\perp \,\&\, B^\perp & (!A)^\perp \overset{\text{def}}{=} ?A^\perp \\
 & \bot^\perp \overset{\text{def}}{=} 1 & \top^\perp \overset{\text{def}}{=} 0 & \\
 & 1^\perp \overset{\text{def}}{=} \bot & 0^\perp \overset{\text{def}}{=} \top &
\end{array}$$

We use the notation $A \multimap B \overset{\text{def}}{=} A^\perp \parr B$. **Rules** are given on Figure 1, where A and B stand for arbitrary formulas, Γ and Δ for sets of (occurrences of) formulas, and $?\Gamma$ for

$$
\frac{}{\vdash X^+, X^-}\,(ax) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,(cut)
$$

$$
\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\,(\parr) \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\,(\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\,(\bot) \qquad \frac{}{\vdash 1}\,(1)
$$

$$
\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\,(\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\,(\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\,(\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\,(\top)
$$

$$
\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\,(?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\,(?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\,(?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,(!)
$$

Figure 1: Rules of Linear Logic

a set of (occurrences of) ?-formulas. Remark we restrict the *ax*-rule to *atomic formulas*, a well-known simplification. This sequent calculus is equipped with a **cut-elimination** procedure, denoted by $\longrightarrow$, that is weakly normalizing and that we recall in Appendix A.

We denote by $\mathrm{Voc}(A)$ the set of atoms occurring in the formula A. For instance, $\mathrm{Voc}((X^+ \parr X^-) \,\&\, Y^-) = \{X, Y\}$. Uniform interpolation can be stated as follows.

Definition 1 (Uniform Interpolant)**.** Given a formula A and an atom X, a **uniform interpolant** for A with respect to X is a formula C such that:

- $\mathrm{Voc}(C) \subseteq \mathrm{Voc}(A) \setminus \{X\}$;
- $\vdash A^\perp, C$ is provable;
- for all formulas B, if $X \notin \mathrm{Voc}(B)$ and $\vdash A^\perp, B$ is provable then $\vdash C^\perp, B$ is provable.

The **uniform interpolation property** holds when every formula admits a uniform interpolant with respect to every atom.

3 Toolbox

We present here two tools that will be of use to prove that our counter-example is indeed a counter-example. The first is a variant of *slices*, the second of *Geometry Of Interaction*.

3.1 A variant of Slices

The &-rule stands apart since it is the sole rule with some sharing of the context Γ. A useful tool when considering &-rules is the notion of a *slice* [Gir87; Gir96] which is a derivation missing some additive component. In a slice, no rule duplicates a context. As our framework has not only additive but also exponential rules, the well-behaved concept is the one of a 0-*slice*, meaning a slice which is not above a !-rule, at depth 0.

Definition 2 (0-slice)**.** For π a derivation, a 0-**slice** of π is a (usually non-derivation) tree π' obtained by deleting one of the two sub-trees of each &-rule of π *that is not above a !-rule.* Thus, in π' some (but maybe not all) &-rules are unary:

$$\dfrac{\vdash A, \Gamma}{\vdash A \,\&\, B, \Gamma}\,(\&_1) \qquad \dfrac{\vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\,(\&_2)$$

Example 3. Here is a derivation followed by its two 0-slices:

$$\dfrac{\dfrac{}{\vdash \top, !(1 \,\&\, 1)}\,(\top) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1 \,\&\, 1}\,(\&)}{\vdash !(1 \,\&\, 1)}\,(!)}{\vdash \bot, !(1 \,\&\, 1)}\,(\bot)}{\vdash \top \,\&\, \bot, !(1 \,\&\, 1)}\,(\&) \qquad \dfrac{\dfrac{}{\vdash \top, !(1 \,\&\, 1)}\,(\top)}{\vdash \top \,\&\, \bot, !(1 \,\&\, 1)}\,(\&_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1 \,\&\, 1}\,(\&)}{\vdash !(1 \,\&\, 1)}\,(!)}{\vdash \bot, !(1 \,\&\, 1)}\,(\bot)}{\vdash \top \,\&\, \bot, !(1 \,\&\, 1)}\,(\&_2)$$

Cut-elimination can be extended in the natural way to 0-slices, with as main difference that the $\& - \oplus$ key case is replaced by:

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \,\&\, B}\,(\&_1) \qquad \dfrac{\vdash A^{\perp}, \Delta}{\vdash A^{\perp} \oplus B^{\perp}, \Delta}\,(\oplus_1)}{\vdash \Gamma, \Delta}\,(cut) \quad \longrightarrow \quad \dfrac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(cut)$$

$$\dfrac{\dfrac{\vdash \Delta, A}{\vdash \Gamma, A \,\&\, B}\,(\&_2) \qquad \dfrac{\vdash A^{\perp}, \Delta}{\vdash B^{\perp} \oplus B^{\perp}, \Delta}\,(\oplus_2)}{\vdash \Gamma, \Delta}\,(cut) \quad \longrightarrow \quad \dfrac{\vdash \Gamma, B \qquad \vdash B^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(cut)$$

Also, it is not possible to eliminate *cut*-rules of the following shapes:

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \,\&\, B}\,(\&_1) \qquad \dfrac{\vdash B^{\perp}, \Delta}{\vdash A^{\perp} \oplus B^{\perp}, \Delta}\,(\oplus_2)}{\vdash \Gamma, \Delta}\,(cut) \qquad \dfrac{\dfrac{\vdash \Delta, A}{\vdash \Gamma, A \,\&\, B}\,(\&_2) \qquad \dfrac{\vdash A^{\perp}, \Delta}{\vdash B^{\perp} \oplus B^{\perp}, \Delta}\,(\oplus_1)}{\vdash \Gamma, \Delta}\,(cut)$$

Lemma 4. *Let π and ϕ be derivations such that $\pi \longrightarrow \phi$. For each* 0*-slice ϕ' of ϕ, there exists a* 0*-slice π' of π such that $\pi' \longrightarrow \phi'$ or $\pi' = \phi'$.*

Proof. We can check that each cut-elimination step respects this property, with the equality case coming from a reduction on rules not in the considered 0-slice. □

Observe a corresponding result for slices does not hold, because a formula of the shape $?(A \oplus B)$ can be associated to both a $\oplus_1$- and a $\oplus_2$-rule. This is a phenomenon at play, for instance, in the Seely isomorphism $?(A \oplus B) \simeq ?A \parr ?B$. The cut-elimination case responsible for this failure is the $?_c - !$ key case, because the two copies of the duplicated derivation may not make the same choices for all of their corresponding &-rules.

3.2 A variant of Geometry Of Interaction

We define here the key tool for proving our counter-example has no uniform interpolant. It falls within the spirit of *Geometry Of Interaction (GOI)* [Gir89]. The quite unusual part is that we will look at *connectives* of the *cut*-formulas along a path from GOI.

Definition 5 (GOI projection graph)**.** The **GOI projection graph** $\mathcal{G}_\pi$ of a derivation (or of a 0-slice) π of conclusion $\vdash \Gamma$ is the undirected edge-bicolored graph defined as follows, colors being *ax* and *cut*.

Vertices: All occurrences of atomic formulas X^+ and X^- (for all atoms) from the formulas of Γ and from the *cut*-formulas A and $A^\perp$ in π, with each *cut*-rule yielding a pair of *cut*-formulas giving separate and dual occurrences.

*ax***-edges:** For each pair of occurrences X^+ and X^- such that π contains an *ax*-rule $\dfrac{}{\vdash X^+, X^-}\,(ax)$, there is an *ax*-edge between X^+ and X^-.[1]

In our figures, *ax*-edges are coloured in **azure** and depicted above the vertices.

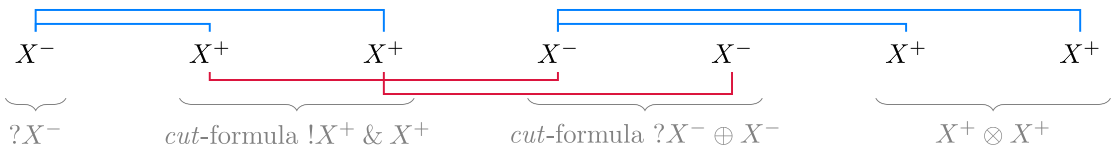
*cut***-edges:** For each *cut*-rule $\dfrac{\vdash \Delta, A \qquad \vdash A^\perp, \Phi}{\vdash \Delta, \Phi}\,(cut)$ in π, there is a *cut*-edge between dual occurrences from the two formulas A and $A^\perp$.

In our figures, *cut*-edges are coloured in **crimson** and depicted below the vertices.

Example 6. Consider the following derivation π of $\vdash ?X^-, X^+ \otimes X^+$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax)}{\vdash ?X^-, X^+}\,(?_d)}{\vdash ?X^-, !X^+}\,(!) \qquad \dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax)}{\vdash ?X^-, X^+}\,(?_d)}{\vdash ?X^-, !X^+ \,\&\, X^+}\,(\&) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax)}{\vdash ?X^-, X^+}\,(?_d) \quad \dfrac{\dfrac{}{\vdash X^-, X^+}\,(ax)}{\vdash ?X^-, X^+}\,(?_d)}{\vdash ?X^-, ?X^-, X^+ \otimes X^+}\,(\otimes)}{\vdash ?X^-, X^+ \otimes X^+}\,(?_c)}{\vdash ?X^- \oplus X^-, X^+ \otimes X^+}\,(\oplus_1)}{\vdash ?X^-, X^+ \otimes X^+}\,(cut)$$

Its GOI projection graph $\mathcal{G}_\pi$ is the following:



Observe they are two ways to obtain several *ax*-edges on a same occurrence: this occurrence can be duplicated using a $?_c$-rule on one of its ancestor (in the formula tree), or by being in the context of a &-rule. Both cases can be observed in Example 6.

[1]This notion extends to general *ax*-rules $\dfrac{}{\vdash A, A^\perp}\,(ax)$ with X^+ occurring in A and X^- in $A^\perp$.

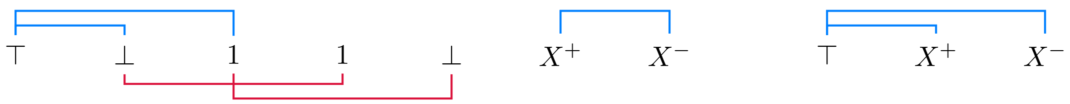
We will consider **alternating** paths in a GOI projection graph, which are paths[2] where consecutive edges are coloured differently—*e.g.* we start with an ax-edge, then take a cut-edge, then an ax-edge, and so on. We have the usual result of GOI about *persistent* paths, *i.e.* some paths are preserved through cut-elimination [DR95]. What is relevant for our purposes is that this holds not only for derivations but also for 0-slices.

Lemma 7. *Consider a cut-elimination step $\pi \longrightarrow \phi$ between two derivations (or between two 0-slices). If there is an alternating path between two vertices u and v in $\mathcal{G}_\phi$, then u and v are also vertices of $\mathcal{G}_\pi$ and there is an alternating path between u and v in $\mathcal{G}_\pi$.*

Proof. By inspection on the kind of the cut-elimination step. As we only care about ax-rules and atoms, the non-immediate cases are either trivial with $\mathcal{G}_\phi \subseteq \mathcal{G}_\pi$ ($\& - \oplus$ key case, $?_w - !$ key case, $\top - cut$ commutative case) or easy to check (ax key case, $?_c - !$ key case, $\& - cut$ commutative case). □

Remark 8. It is easy to find a counter-example to the reciprocal of Lemma 7. One could be tempted to consider not only ax-rules but also $\top$-rules in a GOI projection graph, also adding vertices for all occurrences of units. Unfortunately, such a modified setting does not work for our purposes: Lemma 7 would not hold—consider *e.g.*:

$$\dfrac{\dfrac{}{\vdash \top, \bot \parr 1}\,(\top) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1) \qquad \dfrac{\dfrac{}{\vdash X^+, X^-}\,(ax)}{\vdash \bot, X^+, X^-}\,(\bot)}{\vdash 1 \otimes \bot, X^+, X^-}\,(\otimes)}{\vdash \top, X^+, X^-}\,(cut) \qquad \longrightarrow \qquad \dfrac{}{\vdash \top, X^+, X^-}\,(\top)$$



4 A Counter-Example to Uniform Interpolation

Let us fix two atoms X and Y. Our counter-example is:

$$A \stackrel{\text{def}}{=} X^+ \otimes !\big(X^+ \multimap (X^+ \otimes Y^+)\big) \otimes (X^+ \multimap Y^+) \qquad \begin{cases} B_0 & \stackrel{\text{def}}{=} Y^+ \\ B_{n+1} & \stackrel{\text{def}}{=} B_n \otimes Y^+ \end{cases}$$

The main intuition is the following. The formula A is made of three parts: X^+, a "reusable machine" taking as input X^+ and returning it along with a Y^+, and a "one-use machine" taking as input X^+ and returning Y^+. Thence, A can "produce" any non-null number of Y^+, so that $A \vdash B_n$ is provable for all $n \in \mathbb{N}$. Therefore, one may think that $!Y^+ \otimes Y^+$ is a uniform interpolant for A, and indeed we do have $!Y^+ \otimes Y^+ \vdash B_n$ provable. Nonetheless, $A \vdash !Y^+ \otimes Y^+$ cannot be proved.

[2] A path is a finite alternating sequence of vertices and edges $(v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n)$ such that for all $i \in \{1, \dots, n\}$, the two endpoints of e_i are v_{i-1} and v_i.

We now have to check the three properties claimed in the introduction: $X \notin \mathrm{Voc}\,(B_n)$, and $A \vdash B_n$ is provable for all $n \in \mathbb{N}$, and for all formulas C such that $X \notin \mathrm{Voc}\,(C)$, if $A \vdash C$ and $C \vdash B_n$ are both provable, then C is of size at least n. Note that the first property holds by definition. The second is easy to prove.

Lemma 9. *For all $n \in \mathbb{N}$, there is a derivation of $A \vdash B_n$ in intuitionistic linear logic (hence a derivation of $\vdash A^\perp, B_n$ in classical linear logic).*

Proof. We build a derivation π_n of $X^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n$ by induction on $n \in \mathbb{N}$. The result then follows by exhibiting:

$$\dfrac{\dfrac{\begin{matrix}\pi_n\\ X^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n\end{matrix}}{X^+, !\,(X^+ \multimap (X^+ \otimes Y^+)) \otimes (X^+ \multimap Y^+) \vdash B_n}\,(\otimes \vdash)}{X^+ \otimes !\,(X^+ \multimap (X^+ \otimes Y^+)) \otimes (X^+ \multimap Y^+) \vdash B_n}\,(\otimes \vdash)$$

We define the wanted derivations as:

$$\begin{cases}
\pi_0 \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{}{X^+ \vdash X^+}\,(ax) \qquad \dfrac{}{Y^+ \vdash Y^+}\,(ax)}{X^+, X^+ \multimap Y^+ \vdash Y^+}\,(\multimap \vdash)}{X^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash Y^+}\,(!_w \vdash) \\[3em]
\pi_{n+1} \overset{\text{def}}{=} \dfrac{\dfrac{\dfrac{\dfrac{}{X^+ \vdash X^+}\,(ax) \qquad \dfrac{\dfrac{\begin{matrix}\pi_n\\ X^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n\end{matrix} \qquad \dfrac{}{Y^+ \vdash Y^+}\,(ax)}{X^+, Y^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n \otimes Y^+}\,(\vdash \otimes)}{X^+ \otimes Y^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n \otimes Y^+}\,(\otimes \vdash)}{X^+, X^+ \multimap X^+ \otimes Y^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n \otimes Y^+}\,(\multimap \vdash)}{X^+, !\,(X^+ \multimap X^+ \otimes Y^+)\,, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n \otimes Y^+}\,(!_d \vdash)}{X^+, !\,(X^+ \multimap (X^+ \otimes Y^+))\,, X^+ \multimap Y^+ \vdash B_n \otimes Y^+}\,(!_c \vdash)
\end{cases}$$

□

The third wished property is more complex to prove: we use the tools from Section 3. It also needs an intermediate result, that is not surprising, albeit technical to prove.

Lemma 10. *Let C be a formula such that $X \notin \mathrm{Voc}\,(C)$ and $\vdash A^\perp, C$ is provable. There is a cut-free derivation of $\vdash A^\perp, C$ in which no $!$-rule has $?\,(X^+ \otimes (X^- \parr Y^-))$ in its conclusion sequent.*

Proof. Consider a cut-free derivation π of $\vdash A^\perp, C$ *with a minimal number of $!$-rules having $?\,(X^+ \otimes (X^- \parr Y^-))$ in their conclusion sequents.* We prove this number is null.

By the sub-formula property, any given $!$-rule in π is of the shape

$$\dfrac{\begin{matrix}\pi'\\ \vdash \left(?\,(X^+ \otimes (X^- \parr Y^-))\right)^i, D, ?\Delta\end{matrix}}{\vdash \left(?\,(X^+ \otimes (X^- \parr Y^-))\right)^i, !D, ?\Delta}\,(!)$$

with a conclusion sequent containing $i \geq 0$ copies of $?\,(X^+ \otimes (X^- \parr Y^-))$ for some $i \in \mathbb{N}$, and where $!D, ?\Delta$ are sub-formulas of C. Towards a contradiction, assume $i \geq 1$. Substituting X by 0 in π' yields a derivation $\pi'[0/X]$ of $\vdash (?\,(0 \otimes (\top \parr Y^-)))^i\,, D, ?\Delta$ since, by hypothesis, there is no X in $D, ?\Delta$. Consider any result π'' of fully eliminating *cut*-rules in

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \top, 0 \otimes Y^+}\ (\top)}{\vdash \top \parr (0 \otimes Y^+)}\ (\parr)}{\vdash\, !\,(\top \parr (0 \otimes Y^+))}\ (!) \qquad \dfrac{\begin{matrix}\pi'[0/X] \\ \vdash (?\,(0 \otimes (\top \parr Y^-)))^i\,, D, ?\Delta\end{matrix}}{\vdash\, ?\,(0 \otimes (\top \parr Y^-))\,, D, ?\Delta}\ (?_c)}{\dfrac{\dfrac{\vdash D, ?\Delta}{\vdash\, !D, ?\Delta}\ (!)}{\vdash (?\,(X^+ \otimes (X^- \parr Y^-)))^i\,, !D, ?\Delta}\ (?_w)}\ (cut)$$

where doubled inference rules mean we apply the rule several times depending on i. Then, π'' has no !-rule having $?\,(X^+ \otimes (X^- \parr Y^-))$ in its conclusion sequent by the sub-formula property. Therefore, the derivation obtained from π by replacing the sub-tree $\dfrac{\begin{matrix}\pi' \\ \vdash (?\,(X^+ \otimes (X^- \parr Y^-)))^i\,, D, ?\Delta\end{matrix}}{\vdash (?\,(X^+ \otimes (X^- \parr Y^-)))^i\,, !D, ?\Delta}\ (!)$ with π'' contradicts the minimality assumption on π. ☐

Lemma 11. *Let C be a formula such that $X \notin \mathrm{Voc}\,(C)$. If $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$ are both provable for some number $n \in \mathbb{N}$, then C contains at least $n+1$ occurrences of Y^+.*

Proof. Let us call π and ϕ derivations of respectively $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$. We can assume them *cut*-free. We will use the GOI projection graph of (some 0-slice of) their composition by *cut* to identify Y^-'s in $C^\perp$ linked to the Y^+'s of B_n. By using alternating paths, we will prove these occurrences of Y^- are pairwise distinct, concluding the proof.

<u>*Finding an alternating path in some 0-slice.*</u> Observe that $\dfrac{\begin{matrix}\pi \\ \vdash A^\perp, C\end{matrix} \qquad \begin{matrix}\phi \\ \vdash C^\perp, B_n\end{matrix}}{\vdash A^\perp, B_n}\ (cut)$ reduces to a *cut*-free derivation of $\vdash A^\perp, B_n$, that we call τ. This τ is its own unique 0-slice, since there is no &-connective in $A^\perp, B_n$. By repeated applications of Lemma 4, there is a 0-slice μ of $\dfrac{\begin{matrix}\pi \\ \vdash A^\perp, C\end{matrix} \qquad \begin{matrix}\phi \\ \vdash C^\perp, B_n\end{matrix}}{\vdash A^\perp, B_n}\ (cut)$ that reduces to τ. Consider some occurrence Y^+_α in B_n, with the index α used to distinguish it from other occurrences. Since τ is a *cut*-free derivation of $\vdash A^\perp, B_n$, there must be an *ax*-rule between the occurrence Y^+_α of B_n under consideration and one of the two occurrences of Y^- in $A^\perp$: call it Y^-_1. Hence, there is an alternating path between these two in the GOI projection graph $\mathcal{G}_\tau$ of τ. Then, by repeated applications of Lemma 7, there also is an alternating path p between those occurrences in $\mathcal{G}_\mu$. Thus, there is an occurrence Y^-_β and an *ax*-rule $\dfrac{}{\vdash Y^-_\beta, Y^+_\alpha}\ (ax)$ with Y^-_β–Y^+_α being the last edge of p. This occurrence Y^-_β can only be in

$C^\perp$. We use the alternating path p to show there is no sub-formula $?D$ of $C^\perp$ containing Y^-_β—say differently, there is no ? between the root of the formula $C^\perp$ and Y^-_β. Once this claim is proved, we are done: each occurrence of Y^+ in B_n is linked by an ax-edge to an occurrence of Y^- in $C^\perp$. All these occurrences of Y^- are pairwise distinct: there is no sub-formula $?D$ of $C^\perp$ containing such an occurrence, so they cannot be contracted; and there is no &-rule that can share the identified ax-rules since we are in a 0-slice, and these ax-rules cannot be above a !-rule for there is no ?-connective nor !-connective in B_n. Therefore, we have found $n+1$ occurrences of Y^- in $C^\perp$, as wished.

Exploiting the alternating path. We prove our claim by showing a stronger result:

- for all occurrences of Y^+ of C (resp. of $C^\perp$) belonging to p, there is no ! between the root of C (resp. of $C^\perp$) and these occurrences;
- for all occurrences of Y^- of C (resp. of $C^\perp$) belonging to p, there is no ? between the root of C (resp. of $C^\perp$) and these occurrences.

We proceed by following the edges of p starting from its endpoint Y^-_1 in $A^\perp$ (formally, by induction on the number of edges in p between the considered occurrence and Y^-_1).

Exploiting the alternating path: base case. Consider the first edge Y^-_1–Y^+_2 of p, with Y^+_2 in C. Using Lemma 10, without any loss of generality, there is no !-connective between the root of C and Y^+_2: otherwise, since there is an ax-rule on $\vdash Y^-_1, Y^+_2$, there would be a !-rule (associated to this !-connective) with in its conclusion sequent a ?-sub-formula of $A^\perp$ containing Y^-_1, meaning with $?(X^+ \otimes (X^- \parr Y^-))$.

Exploiting the alternating path: inductive case. See Figure 2 for an illustration of our reasoning. Assume the results holds for a prefix of p, and let us consider its last edge.

1. Suppose this prefix ends with an ax-edge Y^-_k–Y^+_{k+1} with Y^+_{k+1} in C. We know there is no ! between the root of C and Y^+_{k+1}. As p is alternating, the subsequent edge is a cut-edge Y^+_{k+1}–Y^-_{k+2} with Y^-_{k+2} in $C^\perp$. Since Y^+_{k+1} and Y^-_{k+2} are dual occurrences by definition of a cut-edge, there is no ? between the root of $C^\perp$ and Y^-_{k+2}.

2. Suppose this prefix ends with a cut-edge Y^+_k–Y^-_{k+1} with Y^-_{k+1} in $C^\perp$. We know there is no ? between the root of $C^\perp$ and Y^-_{k+1}. As p is alternating, the subsequent edge is an ax-edge Y^-_{k+1}–Y^+_{k+2} with Y^+_{k+2} either in B_n or in $C^\perp$. In the first case, we are done: we have $Y^+_{k+2} = Y^+_\alpha$ since there is no cut-edge with endpoint Y^+_{k+2} to continue our path, and p is supposed to end in Y^+_α. In the second case, since there is an ax-rule $\dfrac{}{\vdash Y^-_{k+1}, Y^+_{k+2}}\,(ax)$ by definition of an ax-edge, it follows there is no ! between the root of $C^\perp$ and Y^+_{k+2}—because the context condition of a corresponding !-rule would prevent the presence of Y^-_{k+1} in all sequents above it.

3. Suppose this prefix ends with an ax-edge Y^-_k–Y^+_{k+1} with Y^+_{k+1} in $C^\perp$. This case is identical to case 1 up to switching C and $C^\perp$.

4. Suppose this prefix ends with a cut-edge Y^+_k–Y^-_{k+1} with Y^-_{k+1} in C. This case is identical to case 2, with no need to consider Y^+_{k+2} in $A^\perp$ since $A^\perp$ has no Y^+.

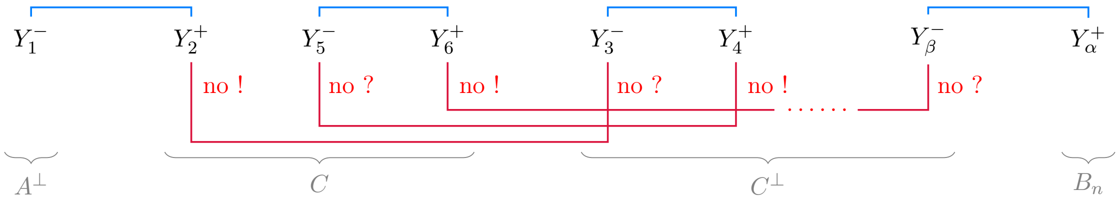


Figure 2: Alternating path in the GOI projection graph $\mathcal{G}_\mu$ in the proof of Lemma 11

This proves our claim, and concludes the proof. □

Remark 12. By going in the other direction of the path p, we can similarly show that:

- for all occurrences of Y^+ of C (resp. of $C^\perp$) belonging to p, there is no ? between the root of C (resp. of $C^\perp$) and these occurrences;
- for all occurrences of Y^- of C (resp. of $C^\perp$) belonging to p, there is no ! between the root of C (resp. of $C^\perp$) and these occurrences.

Proposition 13. *Classical and intuitionistic propositional linear logic do not have the uniform interpolation property.*

Proof. The formula A cannot have a uniform interpolant with respect to X, as such an interpolant would have an unbounded number of occurrences of Y^+ by Lemmas 9 and 11—observing that Lemma 11 implies *a fortiori* its intuitionistic variant. □

5 Conclusion

We presented a formula of (unit-free) multiplicative-exponential linear logic that cannot have a uniform interpolant in linear logic, both classical and intuitionistic. This contrasts sharply with the situation of classical logic and intuitionistic logic, that both have the uniform interpolation property. This counter-example explains why the failure of uniform interpolation for linear logic cannot be transported to classical logic, intuitionistic logic, or multiplicative-additive linear logic. Indeed, we needed in the formula A to interpolate (meaning on the left of the $\vdash$ symbol) both a non-duplicable sub-formula (here X^+) and a duplicable sub-formula (here $!\,(X^+ \multimap (X^+ \otimes Y^+))$), which is impossible in the three aforementioned logics! The formulas we exhibited were not complex, but proving no formula can be an uniform interpolant was quite tedious as it depends heavily on the necessary structure of the formula. Geometry of interaction was a very useful tool to this end—and we do not do how to prove Lemma 11 without using a GOI projection graph.

A natural question following the failure of the uniform interpolation property is the following decision problem: given a formula A, does it have a uniform interpolant? This is an open problem for formulas of (multiplicative-exponential) (intuitionistic) linear logic.

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A Cut-Elimination in Linear Logic

ax	$\dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax) \quad \overset{\pi}{\vdash A,\Gamma}}{\vdash A,\Gamma}\,(cut)$	$\longrightarrow$	$\overset{\pi}{\vdash A,\Gamma}$
$\parr - \otimes - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp,\Gamma}}{\vdash A^\perp \parr B^\perp,\Gamma}\,(\parr) \quad \dfrac{\overset{\phi}{\vdash A,\Delta}\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B,\Delta,\Sigma}\,(\otimes)}{\vdash \Gamma,\Delta,\Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B^\perp,\Gamma}\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A^\perp,\Gamma,\Sigma}\,(cut)\quad \overset{\phi}{\vdash A,\Delta}}{\vdash\Gamma,\Delta,\Sigma}\,(cut)$
$\parr - \otimes - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp,\Gamma}}{\vdash A^\perp \parr B^\perp,\Gamma}\,(\parr) \quad \dfrac{\overset{\phi}{\vdash A,\Delta}\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B,\Delta,\Sigma}\,(\otimes)}{\vdash \Gamma,\Delta,\Sigma}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,B^\perp,\Gamma}\quad \overset{\phi}{\vdash A,\Delta}}{\vdash B^\perp,\Gamma,\Sigma}\,(cut)\quad \overset{\tau}{\vdash B,\Sigma}}{\vdash\Gamma,\Delta,\Sigma}\,(cut)$
$\bot - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash\Gamma}}{\vdash\bot,\Gamma}\,(\bot)\quad \dfrac{}{\vdash 1}\,(1)}{\vdash\Gamma}\,(cut)$	$\longrightarrow$	$\overset{\pi}{\vdash\Gamma}$
$\& - \oplus_1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}\quad\overset{\phi}{\vdash B^\perp,\Gamma}}{\vdash A^\perp \,\&\, B^\perp,\Gamma}\,(\&)\quad \dfrac{\overset{\tau}{\vdash A,\Delta}}{\vdash A\oplus B,\Delta}\,(\oplus_1)}{\vdash\Gamma,\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\phi}{\vdash A^\perp,\Gamma}\quad\overset{\tau}{\vdash A,\Delta}}{\vdash\Gamma,\Delta}\,(cut)$
$\& - \oplus_2$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}\quad\overset{\phi}{\vdash B^\perp,\Gamma}}{\vdash A^\perp \,\&\, B^\perp,\Gamma}\,(\&)\quad \dfrac{\overset{\tau}{\vdash B,\Delta}}{\vdash A\oplus B,\Delta}\,(\oplus_2)}{\vdash\Gamma,\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash B^\perp,\Gamma}\quad\overset{\tau}{\vdash B,\Delta}}{\vdash\Gamma,\Delta}\,(cut)$
$?d - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}}{\vdash ?A^\perp,\Gamma}\,(?d)\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash A^\perp,\Gamma}\quad\overset{\phi}{\vdash A,?\Delta}}{\vdash\Gamma,?\Delta}\,(cut)$
$?c - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp,?A^\perp,\Gamma}}{\vdash ?A^\perp,\Gamma}\,(?c)\quad \dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp,?A^\perp,\Gamma}\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash ?A^\perp,\Gamma,?\Delta}\,(cut)\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta,?\Delta}\,(cut)}{\vdash\Gamma,?\Delta}\,(?c)$
$?w - !$	$\dfrac{\dfrac{\overset{\pi}{\vdash\Gamma}}{\vdash ?A^\perp,\Gamma}\,(?w)\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}\,(!)}{\vdash\Gamma,?\Delta}\,(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash\Gamma}}{\vdash\Gamma,?\Delta}\,(?w)$

Table 1: Cut-elimination – Key cases

$cut - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B^{\perp},\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B^{\perp},\Gamma,\Delta}(cut)\quad\overset{\tau}{\vdash B,\Sigma}}{\vdash\Gamma,\Delta,\Sigma}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B^{\perp},\Gamma}\quad\overset{\tau}{\vdash B,\Sigma}}{\vdash A^{\perp},\Gamma,\Sigma}(cut)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash\Gamma,\Delta,\Sigma}(cut)$
$\parr - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,C,\Gamma}}{\vdash A^{\perp},B\parr C,\Gamma}(\parr)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B\parr C,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,C,\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B,C,\Gamma,\Delta}(cut)}{\vdash B\parr C,\Gamma,\Delta}(\parr)$
$\otimes - cut - 1$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}\quad\overset{\phi}{\vdash C,\Delta}}{\vdash A^{\perp},B\otimes C,\Gamma,\Delta}(\otimes)\quad\overset{\tau}{\vdash A,\Sigma}}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}\quad\overset{\tau}{\vdash A,\Sigma}}{\vdash B,\Gamma,\Sigma}(cut)\quad\overset{\phi}{\vdash C,\Delta}}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}(\otimes)$
$\otimes - cut - 2$	$\dfrac{\dfrac{\overset{\pi}{\vdash B,\Gamma}\quad\overset{\phi}{\vdash A^{\perp},C,\Delta}}{\vdash A^{\perp},B\otimes C,\Gamma,\Delta}(\otimes)\quad\overset{\tau}{\vdash A,\Sigma}}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}(cut)$	$\longrightarrow$	$\dfrac{\overset{\pi}{\vdash B,\Gamma}\quad\dfrac{\overset{\phi}{\vdash A^{\perp},C,\Delta}\quad\overset{\tau}{\vdash A,\Sigma}}{\vdash C,\Delta,\Sigma}(cut)}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}(\otimes)$
$\bot - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},\Gamma}}{\vdash A^{\perp},\bot,\Gamma}(\bot)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash\bot,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash\Gamma,\Delta}(cut)}{\vdash\bot,\Gamma,\Delta}(\bot)$
$\& - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}\quad\overset{\phi}{\vdash A^{\perp},C,\Gamma}}{\vdash A^{\perp},B\,\&\,C,\Gamma}(\&)\quad\overset{\tau}{\vdash A,\Delta}}{\vdash B\,\&\,C,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}\quad\overset{\tau}{\vdash A,\Delta}}{\vdash B,\Gamma,\Delta}(cut)\quad\dfrac{\overset{\phi}{\vdash A^{\perp},C,\Gamma}\quad\overset{\tau}{\vdash A,\Delta}}{\vdash C,\Gamma,\Delta}(cut)}{\vdash B\,\&\,C,\Gamma,\Delta}(\&)$
$\oplus_1 - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}}{\vdash A^{\perp},B\oplus C,\Gamma}(\oplus_1)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B\oplus C,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B,\Gamma,\Delta}(cut)}{\vdash B\oplus C,\Gamma,\Delta}(\oplus_1)$
$\oplus_2 - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},C,\Gamma}}{\vdash A^{\perp},B\oplus C,\Gamma}(\oplus_2)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B\oplus C,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},C,\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash C,\Gamma,\Delta}(cut)}{\vdash B\oplus C,\Gamma,\Delta}(\oplus_2)$
$\top - cut$	$\dfrac{\dfrac{}{\vdash A^{\perp},\top,\Gamma}(\top)\quad\overset{\pi}{\vdash A,\Delta}}{\vdash\top,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{}{\vdash\top,\Gamma,\Delta}(\top)$
$?d - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}}{\vdash A^{\perp},?B,\Gamma}(?d)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash ?B,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},B,\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash B,\Gamma,\Delta}(cut)}{\vdash ?B,\Gamma,\Delta}(?d)$
$?c - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},?B,?B,\Gamma}}{\vdash A^{\perp},?B,\Gamma}(?c)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash ?B,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},?B,?B,\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash ?B,?B,\Gamma,\Delta}(cut)}{\vdash ?B,\Gamma,\Delta}(?c)$
$?w - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},\Gamma}}{\vdash A^{\perp},?B,\Gamma}(?w)\quad\overset{\phi}{\vdash A,\Delta}}{\vdash ?B,\Gamma,\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp},\Gamma}\quad\overset{\phi}{\vdash A,\Delta}}{\vdash\Gamma,\Delta}(cut)}{\vdash ?B,\Gamma,\Delta}(?w)$
$! - cut$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp},B,?\Gamma}}{\vdash ?A^{\perp},!B,?\Gamma}(!)\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}(!)}{\vdash !B,?\Gamma,?\Delta}(cut)$	$\longrightarrow$	$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^{\perp},B,?\Gamma}\quad\dfrac{\overset{\phi}{\vdash A,?\Delta}}{\vdash !A,?\Delta}(!)}{\vdash B,?\Gamma,?\Delta}(cut)}{\vdash !B,?\Gamma,?\Delta}(!)$

Table 2: Cut-elimination – Commutative cases