Bayesian Networks and Proof-Nets: the proof-theory of Bayesian Inference

Rémi Di Guardia, Thomas Ehrhard, Jérôme Evrard, Claudia Faggian

IRIF (CNRS, Université Paris Cité)

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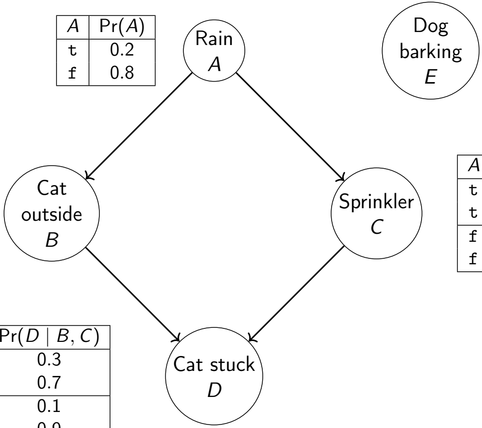
Plan

- Bayesian Networks
- Bayesian Networks in Proof Nets of Linear Logic
- Example of Graphical Reasoning

Bayesian Networks

Bayesian Network = DAG + a (conditional) probability per vertex



A	$\Pr(A)$
t	0.2
f	0.8

E	$\Pr(E)$
t	0.7
f	0.3

A	B	$\Pr(B \mid A)$
t	t	0.4
t	f	0.6
f	t	0.7
f	f	0.3

A	C	$\Pr(C \mid A)$
t	t	0.3
t	f	0.7
f	t	0.9
f	f	0.1

B	C	D	$\Pr(D \mid B, C)$
t	t	t	0.3
t	t	f	0.7
t	f	t	0.1
t	f	f	0.9
f	t	t	0
f	t	f	1
f	f	t	0
f	f	f	1

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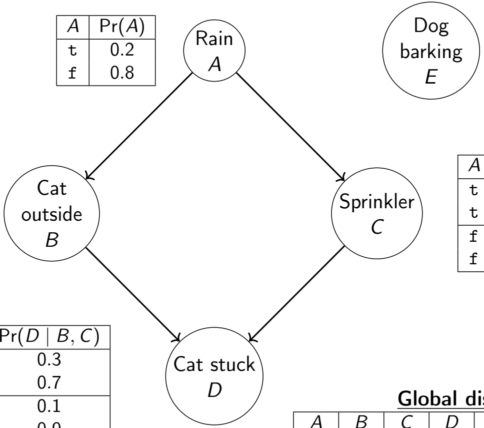

A	Pr(A)
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t	t	t	0.3
t	t	f	0.7
t	f	t	0.1
t	f	f	0.9
f	t	t	0
f	t	f	1
f	f	t	0
f	f	f	1

Global distribution

A	B	C	D	E	Pr(A, B, C, D, E)
t	t	t	t	t	0.00504
t	t	t	t	f	0.00216
t	t	t	f	t	0.01176
t	t	t	f	f	0.00504
...	...	...	...	...	...

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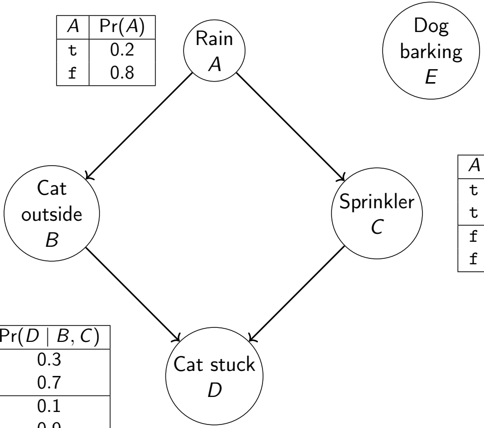

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Query

$\Pr(D = \mathtt{t}, E = \mathtt{t} \mid A = \mathtt{t})$

$\longrightarrow$ structure allows *efficient* computation

Bayesian Networks

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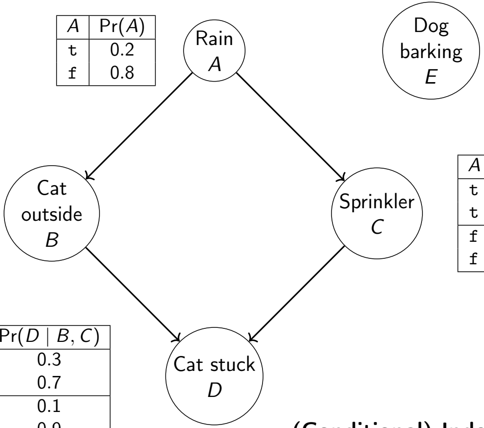

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f	t	f	1
f	f	t	0
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(Conditional) Independence

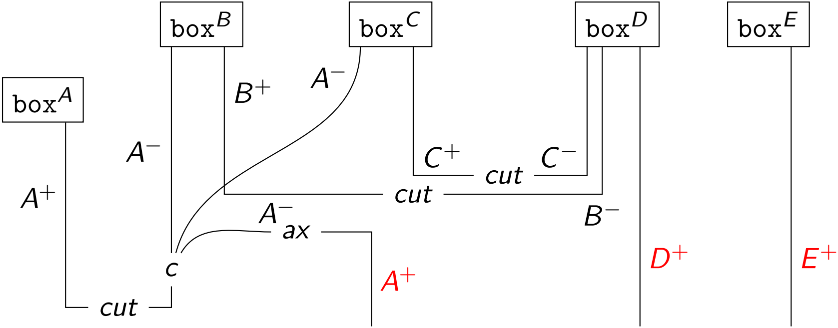
$\Pr(A = \mathtt{t}, E = \mathtt{t}) = \Pr(A = \mathtt{t}) \Pr(E = \mathtt{t})$

$\Pr(B = \mathtt{f}, C = \mathtt{t} \mid A = \mathtt{f}) =$
$\qquad \Pr(B = \mathtt{f} \mid A = \mathtt{t}) \Pr(C = \mathtt{t} \mid A = \mathtt{t})$

$\longrightarrow$ deducible directly from the *DAG*

Bayesian Proof Nets

Bayesian Networks can be embedded in Proof Nets



Query in the syntax = labels of the pending edges
$\longrightarrow$ here the semantics of the proof net P is $(\!|P|\!) = \Pr(A, D, E)$

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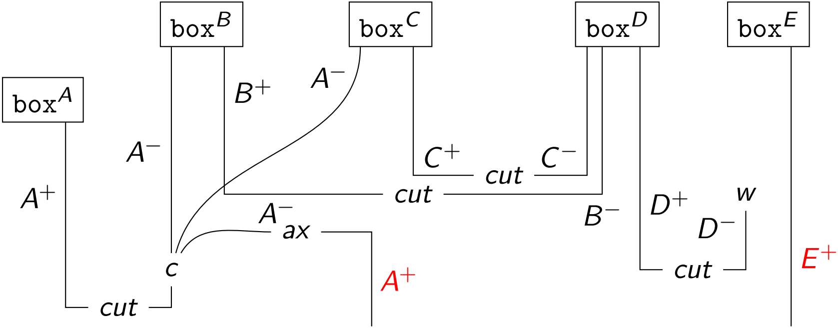
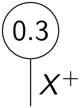


Query in the syntax = labels of the pending edges
$\longrightarrow$ here the semantics of the proof net P is $(\!|P|\!) = \Pr(A, E)$

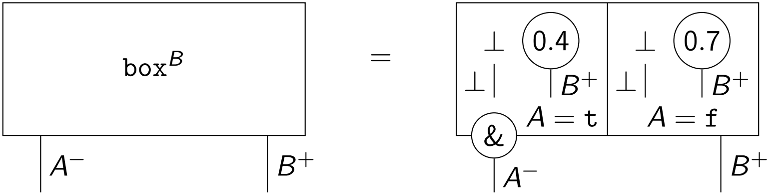
What is inside a box?

Secretly, variables $A^+, B^+, \ldots$ are not atoms but **booleans** $(= 1 \oplus 1)$.

A box = **Bernoullis** + **"if . . . then . . . else . . ."** $(= \&)$.



X	$\Pr(X)$
t	0.3
f	0.7



A	B	$\Pr(B \mid A)$
t	t	0.4
t	f	0.6
f	t	0.7
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Conditional Independence Graphically

Definition

Variables X and Y are **conditionally independent** given Z if $\Pr(X, Y \mid Z) = \Pr(X \mid Z) \Pr(Y \mid Z)$.

Definition

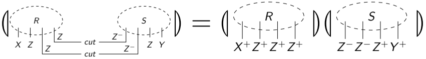
In a Bayesian Proof Net, X and Y are **disconnected** by Z if there is no path between $\mathtt{box}^X$ and $\mathtt{box}^Y$ once removing all edges labeled by Z.

Theorem

If X and Y are disconnected by Z then X and Y are conditionally independent given Z in the associated distribution.

Graphical Proof

$\Pr(X, Y, Z) =$



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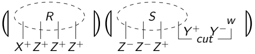
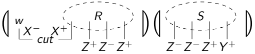
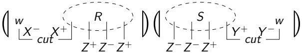


$\Pr(Y, Z) =$



$\Pr(Z) =$



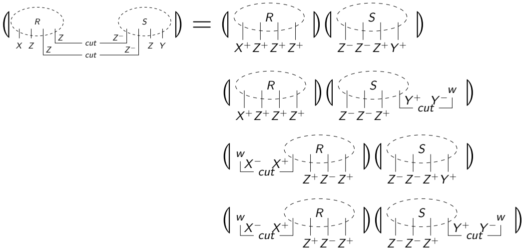
Graphical Proof

$$\Pr(X, Y, Z) = \quad = $$

$$\Pr(X, Z) = $$

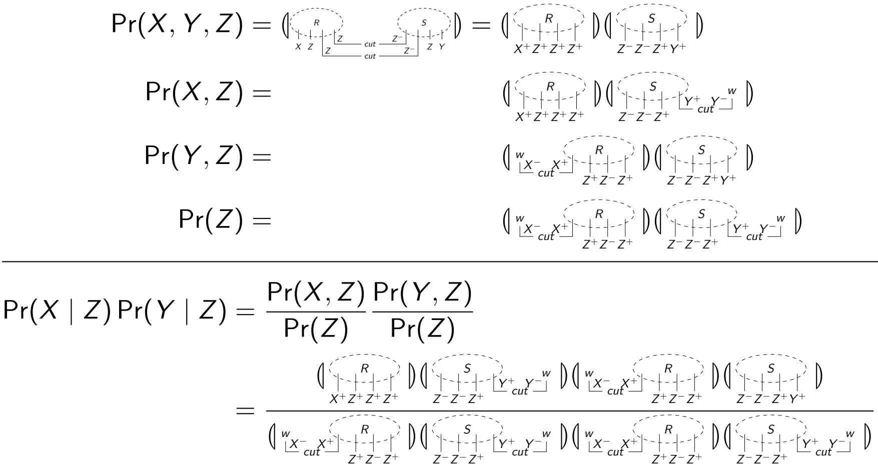
$$\Pr(Y, Z) = $$

$$\Pr(Z) = $$

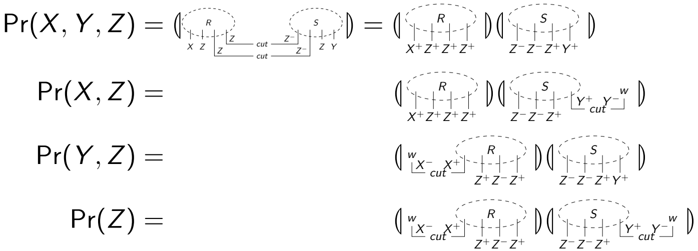


$$\Pr(X \mid Z) \Pr(Y \mid Z) = \frac{\Pr(X, Z)}{\Pr(Z)} \frac{\Pr(Y, Z)}{\Pr(Z)}$$

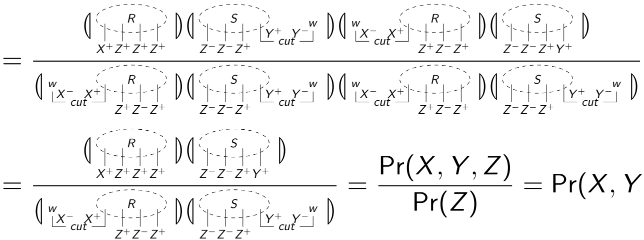
Graphical Proof



Graphical Proof



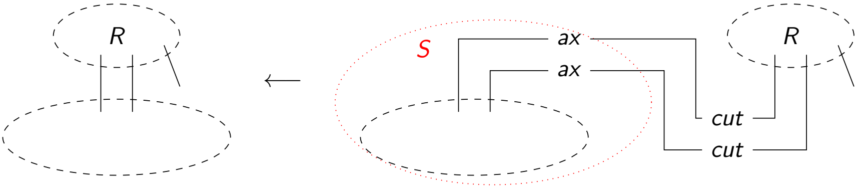
$$\Pr(X \mid Z)\Pr(Y \mid Z) = \frac{\Pr(X,Z)}{\Pr(Z)}\frac{\Pr(Y,Z)}{\Pr(Z)}$$



$$= \frac{\Pr(X,Y,Z)}{\Pr(Z)} = \Pr(X, Y \mid Z)$$

. . . and there is more!

- Can transfer computations algorithms (*e.g. Variable Elimination*) in Proof Nets, using the **usual rewriting rules** (cut-elimination).



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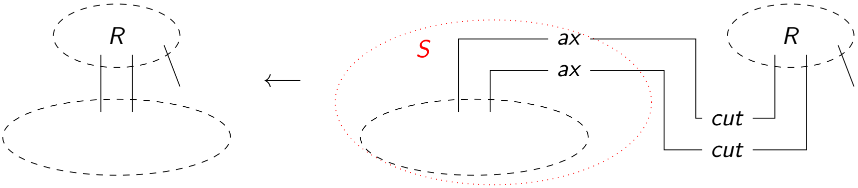

- **Quantum** Bayesian Networks exist, and can be studied in Proof Nets too! *(work in progress)*

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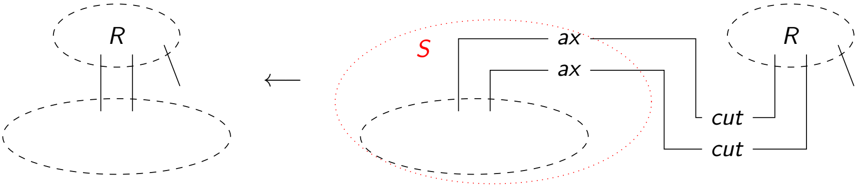

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Thank you!