Semantics and types for quantum Bayesian networks

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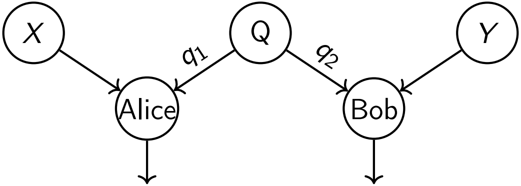
Introduction – Bell's Experiment



Question: What is $Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$?

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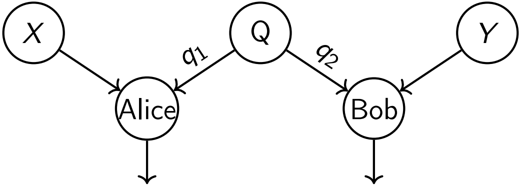


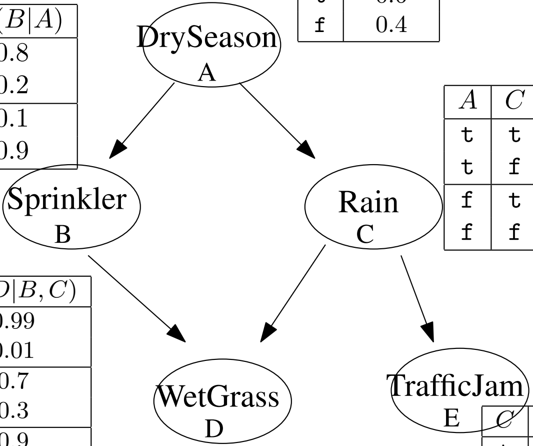
Question: What is $Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$?

General Problem: Taking a quantum system, what is the probability of obtaining a given measurement?

Classical case – Bayesian Networks

A	$Pr(A)$
t	0.6
f	0.4

A	B	$Pr(B\|A)$
t	t	0.8
t	f	0.2
f	t	0.1
f	f	0.9



A	C	$Pr(C\|A)$
t	t	0.2
t	f	0.8
f	t	0.75
f	f	0.25

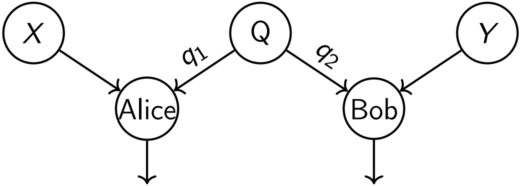
C	B	D	$Pr(D\|B,C)$
t	t	t	0.99
t	t	f	0.01
t	f	t	0.7
t	f	f	0.3
f	t	t	0.9
f	t	f	0.1
f	f	t	0.01
f	f	f	0.99

C	E	$Pr(E\|C)$
t	t	0.7
t	f	0.3
f	t	0.1
f	f	0.9

Quantum Bayesian Networks

Definition: Quantum Bayesian Networks [HLP14]

DAG with a **quantum instrument** per node such that composing these instruments yields a probability distribution.



$$\begin{aligned} Q &= \textit{Preparation of the entangled } q_1 \textit{ and } q_2 \\ Alice &= \textit{Measure on } q_1 \textit{ parameterized by the value of } X \end{aligned}$$

Limits

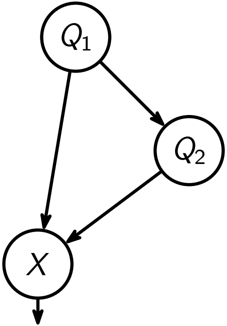
Problems: two wished results we do not have

- **Compositionality:** Given a decomposition of a network, is the data of the full network obtained from the data of each part?
  $\longrightarrow$ a big advantage of a graphical syntax, and a main property of Bayesian networks

- **Modularity:** Given two parts, can they compose to give a network?
  $\longrightarrow$ the result must be a DAG

Our contributions:
Solutions for these two problems, by giving another presentation of Quantum Bayesian Networks

Compositionality Limit

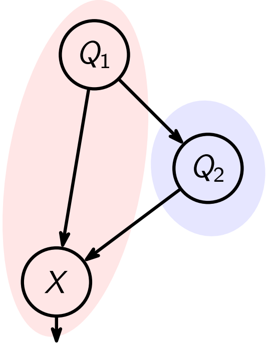


$$\phi^{Q_1} : \mathbb{C} \to H_1 \otimes H_2$$
$$\phi^{Q_2} : H_2 \to H_3$$
$$\phi^{X}(x) : H_1 \otimes H_3 \to \mathbb{C}$$

Whole graph:

$$\begin{aligned}\phi^{Q_1,Q_2,X}(x) &: \mathbb{C} \to \mathbb{C} \\ &= \phi^{X}(x) \circ (id_{H_1} \otimes \phi^{Q_2}) \circ \phi^{Q_1}\end{aligned}$$

Compositionality Limit



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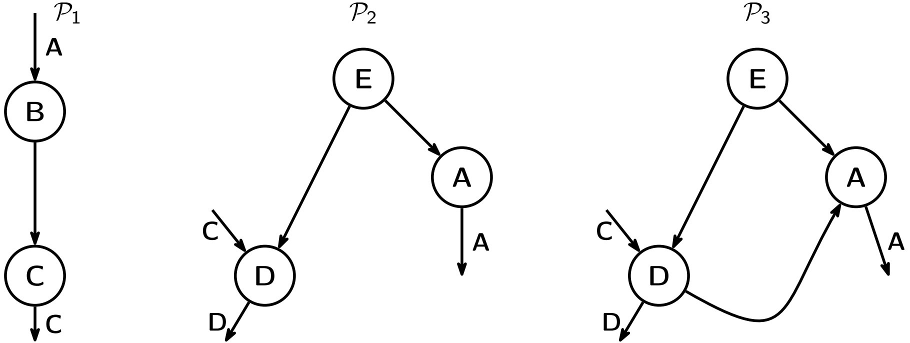
$\phi^{Q_1,X}(x)$: $H_3 \to H_2$ $\qquad$ ϕ^{Q_2}: $H_2 \to H_3$

$\Longrightarrow \phi^{Q_1,X,Q_2}(x)$ cannot simply be the **composition** of $\phi^{Q_1,X}(x)$ and ϕ^{Q_2}!
(gives either $H_3 \to H_3$ or $H_2 \to H_2$)

Idea: Functions do not work well at graph level, matrices would be better
(also in Bayesian networks: factors instead of conditional probabilities)

Modularity Limit



- Part $\mathcal{P}_1$ waits for input A and outputs C
- Parts $\mathcal{P}_2$ and $\mathcal{P}_3$ wait for input C and output A and D

Question: Is it "legal" to branch $\mathcal{P}_1$ to $\mathcal{P}_2$? $\mathcal{P}_1$ to $\mathcal{P}_3$?

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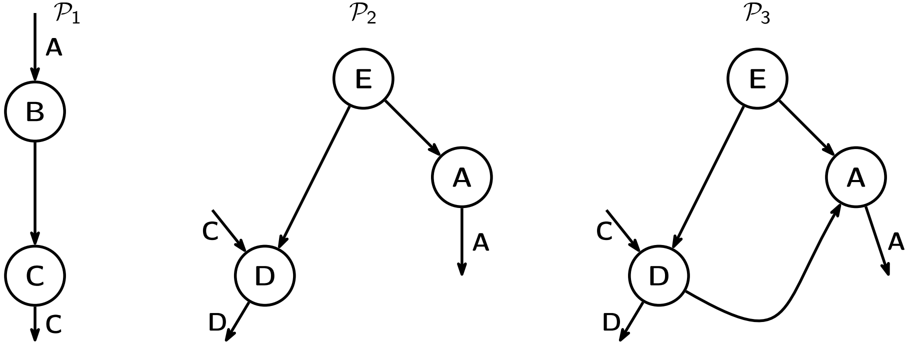

- Part $\mathcal{P}_1$ waits for input A and outputs C
- Parts $\mathcal{P}_2$ and $\mathcal{P}_3$ wait for input C and output A and D

Question: Is it "legal" to branch $\mathcal{P}_1$ to $\mathcal{P}_2$? $\mathcal{P}_1$ to $\mathcal{P}_3$?
$\longrightarrow$ $\mathcal{P}_1 \cup \mathcal{P}_2$ is a QBN but $\mathcal{P}_1 \cup \mathcal{P}_3$ is *not* (cycle $A \to B \to C \to D \to A$)

Idea: Inputs & outputs are insufficient, we need a **type**

Plan

- Compositionality by Quantum Factors

- Modularity by Typing

Quantum Factors

Instead of associating a quantum instrument to a node associate a:

Definition: Quantum Factor

Take random variables $\mathbb{X} = (X_1, \ldots, X_n)$, and Hilbert spaces $\mathbb{H} = (H_1, \ldots, H_m)$. A **Quantum Factor** on $(\mathbb{X}, \mathbb{H})$ is a function ϕ from $Val(\mathbb{X}) = \prod_{i=1}^{n} Val(X_i)$ to *positive* matrices in $\bigotimes_{j=1}^{m} H_j$.

Equipped with a **product** $\odot$, such that for ϕ_1 and ϕ_2 respectively on $(\mathbb{X}_1, \mathbb{H}_1)$ and $(\mathbb{X}_2, \mathbb{H}_2)$, $\phi_1 \odot \phi_2$ is on $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{H}_1 \, \Delta \, \mathbb{H}_2)$.

Only quantum: get $\otimes$-networks and their contraction
Only classical: get factors from Bayesian networks and their product

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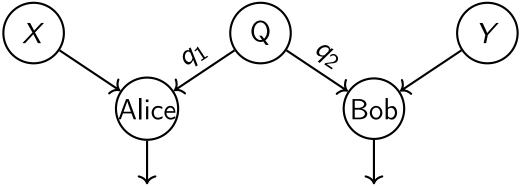
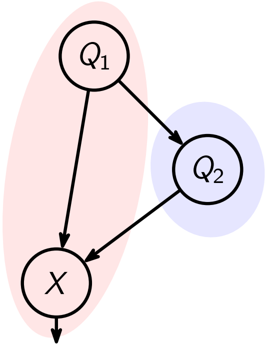
quantum instrument $\rightarrow$ **quantum factor:** pre-compose by cap



Compositionality with quantum factors



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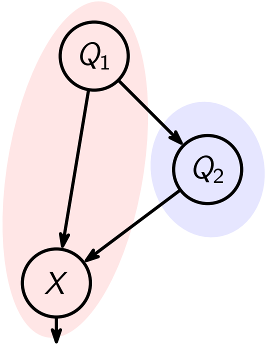

$$\phi^{Q_1} : H_1 \otimes H_2$$
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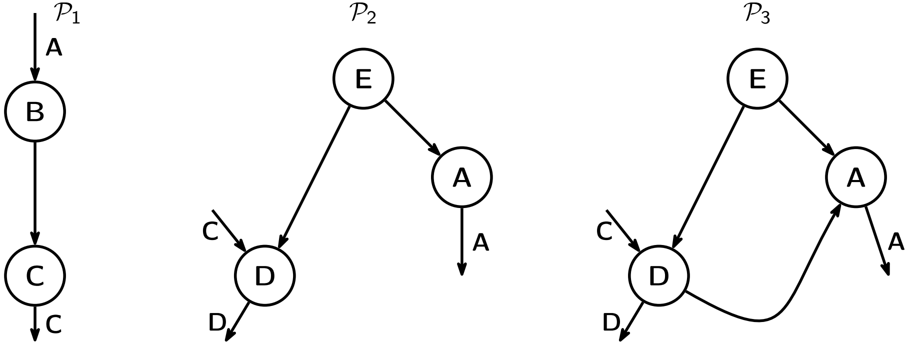
$\phi^{Q_1,X} : Val(X) \to H_3 \otimes H_2$ $\qquad$ $\phi^{Q_2} : H_2 \otimes H_3$

$\Longrightarrow \phi^{Q_1,X,Q_2}(x) = \phi^{Q_1,X} \odot \phi^{Q_2}$

More generally, can compute for any order on the nodes!

Modularity



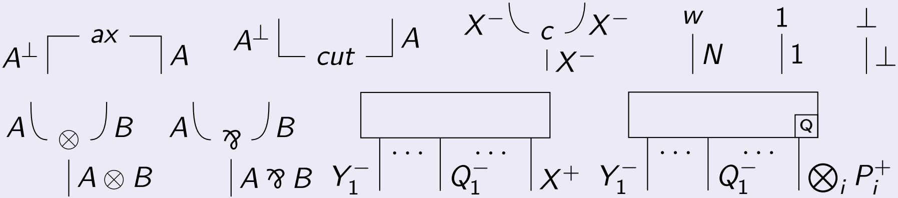
Observation: $\mathcal{P}_1 \cup \mathcal{P}_2$ is a QBN but not $\mathcal{P}_1 \cup \mathcal{P}_3$

How to ensure two parts always form a QBN?
We add a type = an interface
Could do so by patching QBN and getting yet another new syntax...
We prefer to use **proof-nets**, graphs from linear logic adapted to typing

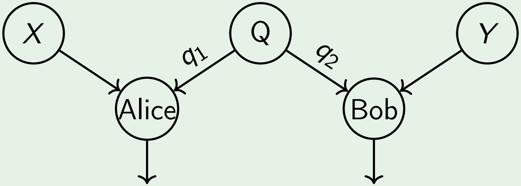
Proof-Nets for Quantum Bayesian Networks

Definition: Proof-Net

A graph respecting some *graphical criterion* and built from:



Example



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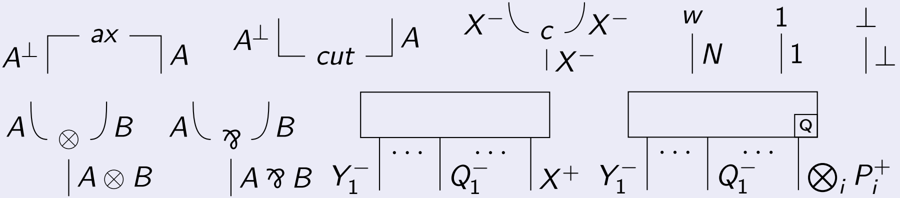

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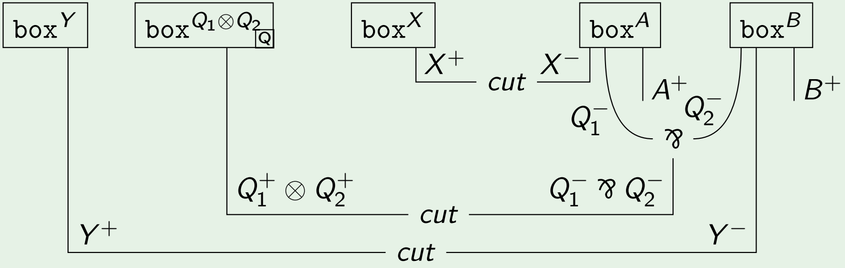
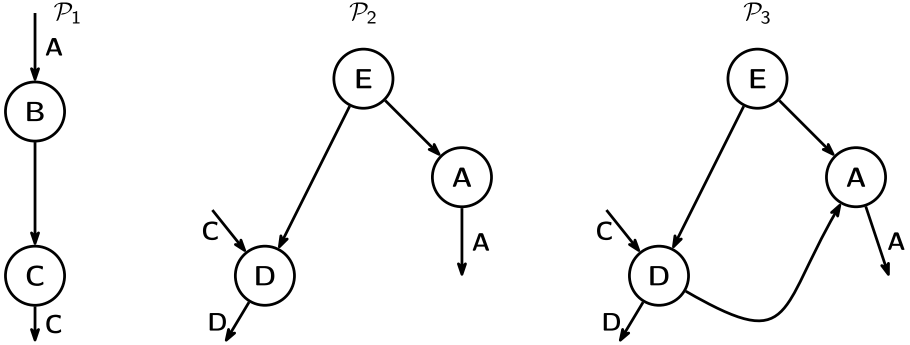


Modularity with types



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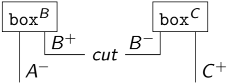
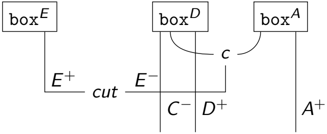
Modularity with types

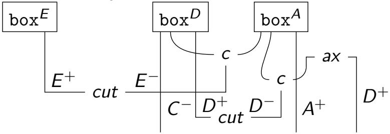
$\mathcal{P}_1$



$\mathcal{P}_2$



$\mathcal{P}_3$



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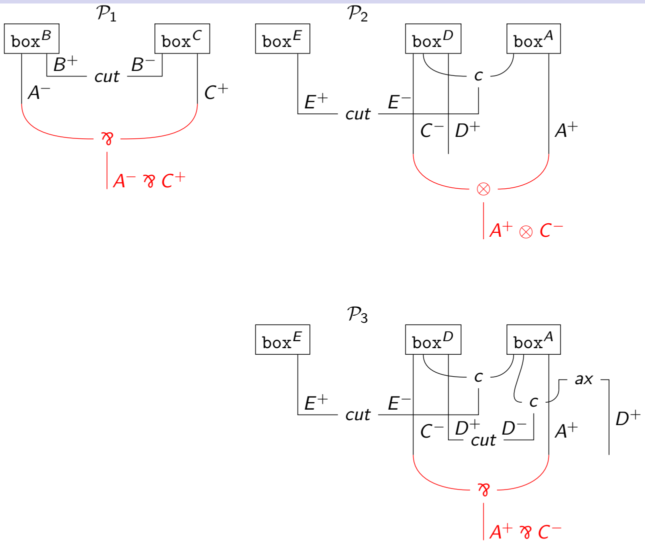

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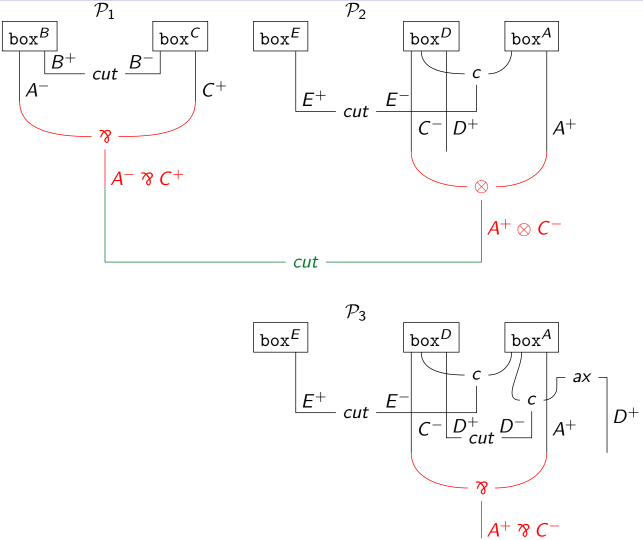

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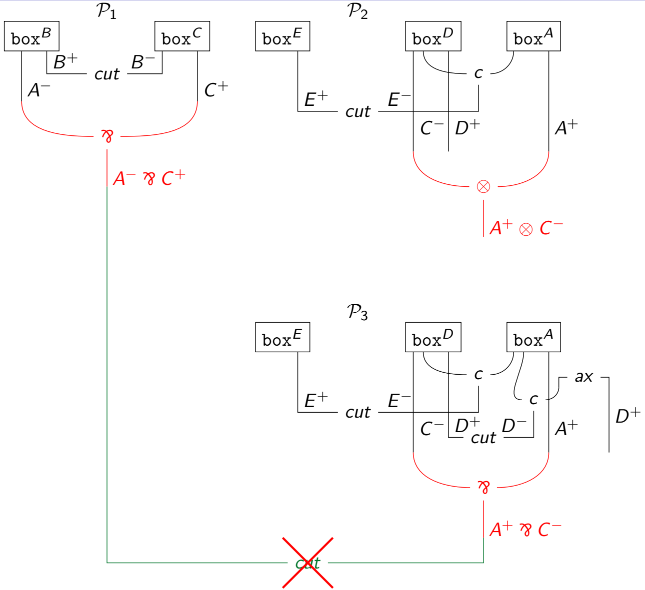

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Conclusion

Contributions

- Compositionality by modifying the semantic:
  from quantum instruments to (more general) *quantum factors*
- Modularity by typing in proof-nets:
  proof-theoretic approach adding an *interface*, parts with compatible interfaces are those giving a QBN

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Perspectives

- Compositionality can be used to study *conditional independence (no-signaling)* between random variables, even with quantum causes
- Modularity allows a weak form of *higher-order* (linear application), can we do more?
- Proof-Nets have *rewriting rules* (cut-elimination) corresponding to computations: can they be used to compute efficiently the quantum factor of the full network? (as in the classical case)

Thank you for
your attention!

References I

[HLP14] Joe Henson, Raymond Lal, and Matthew F Pusey. "Theory-independent limits on correlations from generalized Bayesian networks". In: *New Journal of Physics* 16.11 (Nov. 2014). ISSN: 1367-2630. DOI: 10.1088/1367-2630/16/11/113043.