Quantum Bayesian Networks: Compositionality and Typing via Linear Logic

Rémi Di Guardia, Thomas Ehrhard, Claudia Faggian

IRIF – CNRS, Inria, Université Paris Cité

LIQCS, 18 June 2026

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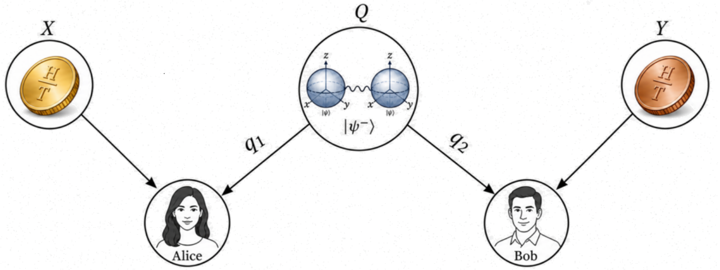
Quantum – Bell's Experiment



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\begin{aligned}
Q &= \textit{Preparation of the entangled } q_1 \textit{ and } q_2 \\
X &= \textit{Flip a coin} \\
Alice &= \textit{Measure on } q_1 \textit{ parameterized by the value of } X
\end{aligned}
$$

Question: What is $\Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$?

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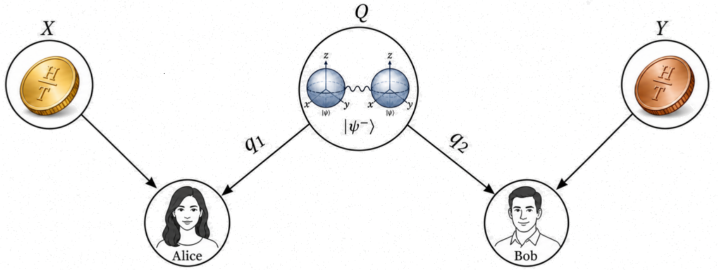

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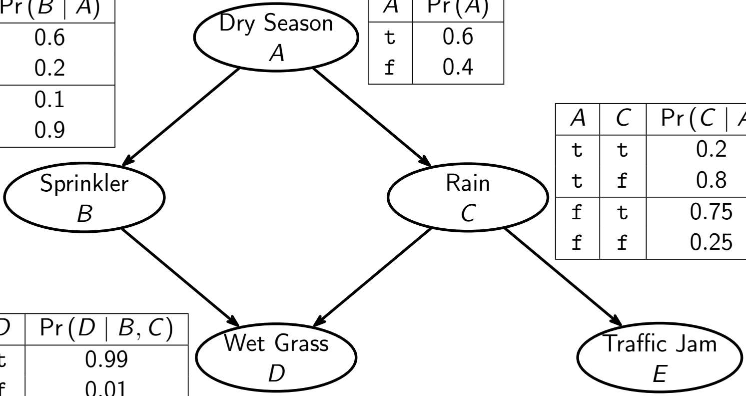
Question: What is $\Pr(Alice = 0, Bob = 0 \mid X = 1, Y = 0)$?

General Problem: Considering a quantum system, what is the probability of obtaining a given measurement?

Bayesian Networks

A	B	$\Pr(B \mid A)$
t	t	0.6
t	f	0.2
f	t	0.1
f	f	0.9

A	$\Pr(A)$
t	0.6
f	0.4



A	C	$\Pr(C \mid A)$
t	t	0.2
t	f	0.8
f	t	0.75
f	f	0.25

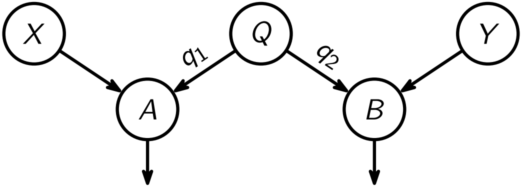
B	C	D	$\Pr(D \mid B, C)$
t	t	t	0.99
t	t	f	0.01
t	f	t	0.7
t	f	f	0.3
f	t	t	0.9
f	t	f	0.1
f	f	t	0.01
f	f	f	0.99

C	E	$\Pr(E \mid C)$
t	t	0.7
t	f	0.3
f	t	0.1
f	f	0.9

Quantum Bayesian Networks

Definition: Quantum Bayesian Networks [Henson, Lal, Pusey 2014]

DAG with a family of **quantum operations** per node such that composing these top-down yields a *probability distribution*.



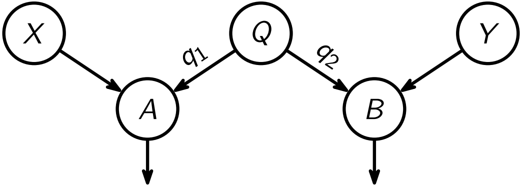
Q	:	$\mathbb{C} \to \mathcal{H}_1 \otimes \mathcal{H}_2$
	$=$	Preparation of the entangled q_1 and q_2
X	:	a map $f_x : \mathbb{C} \to \mathbb{C}$ for each value x of X (with $\sum_x f_x(1) = 1$)
	$=$	Flip a coin
A	:	a map $g_{x,a} : \mathcal{H}_1 \to \mathbb{C}$ for each value x of X and a of A (with $\sum_a g_{x,a}(q_1) = 1$)
	$=$	Measure on q_1 parameterized by the value of X

Uses

Motivations: study probabilities of systems both quantum & classical

- **Conditional Independence / No-Signaling**
  When are *classical* data independent? conditionally independent?
  $\longrightarrow$ *d-separation*, a purely graphical criterion
  Example: A and Y independent in Bell's experiment, but A and Y not independent given B

- **In which situations is quantum stronger than classical?**
  Well-known that in Bell's Experiment quantum $>$ classical
  $\longrightarrow$ DAGs where quantum $>$ classical? where quantum $=$ classical?



Limits

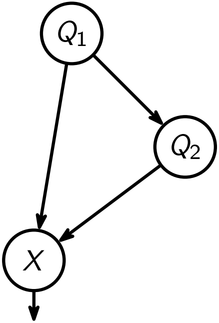
Problem: lack of two properties

- **Compositionality**
  Given a *decomposition* of a network, is the data of the full network obtained from the data of each *part*?
  $\longrightarrow$ a big advantage of a *graphical* syntax, and a main property of Bayesian networks

- **Modularity**
  Given two *parts*, do they *compose* into a network?
  $\longrightarrow$ the result must be a *DAG*

Our contributions:
Solutions for these two problems, by giving another presentation of Quantum Bayesian Networks

Limits – Compositionality

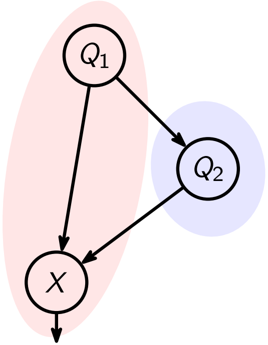


$$\phi^{Q_1} : \mathbb{C} \to \mathcal{H}_1 \otimes \mathcal{H}_2$$
$$\phi^{Q_2} : \mathcal{H}_2 \to \mathcal{H}_3$$
$$\phi^{X}(x) : \mathcal{H}_1 \otimes \mathcal{H}_3 \to \mathbb{C}$$

Full network:

$$\phi^{Q_1,Q_2,X}(x) : \mathbb{C} \to \mathbb{C}$$
$$= \phi^{X}(x) \circ (id_{\mathcal{H}_1} \otimes \phi^{Q_2}) \circ \phi^{Q_1}$$

Limits – Compositionality



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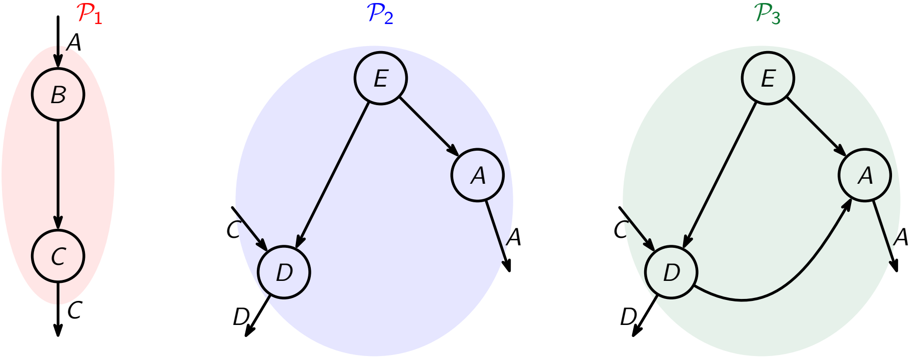
$\phi^{Q_1,X}(x) : \mathcal{H}_3 \to \mathcal{H}_2$ $\phi^{Q_2} : \mathcal{H}_2 \to \mathcal{H}_3$

$\implies \phi^{Q_1,Q_2,X}(x)$ is not the **composition** of $\phi^{Q_1,X}(x)$ and ϕ^{Q_2}!

(typed either $\mathcal{H}_3 \to \mathcal{H}_3$ or $\mathcal{H}_2 \to \mathcal{H}_2$)

Idea: **Functions** do not work well at graph level, **matrices** would be better (also in Bayesian networks: factors instead of conditional probabilities)

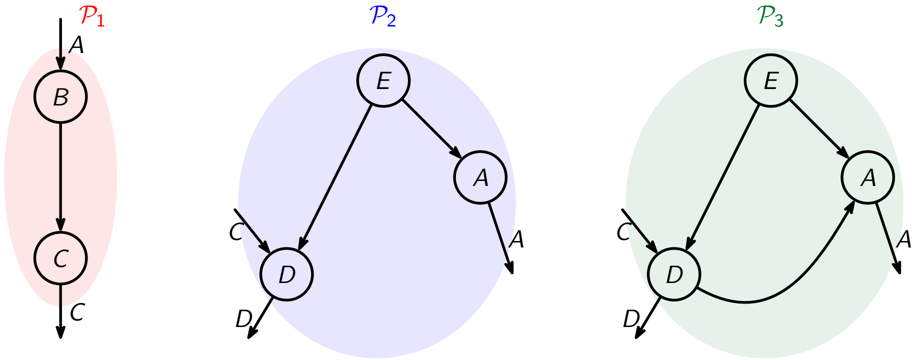
Limits – Modularity



- Part $\mathcal{P}_1$ waits for input A and outputs C
- Parts $\mathcal{P}_2$ and $\mathcal{P}_3$ wait for input C and output A and D

Is it "legal" to branch $\mathcal{P}_1$ to $\mathcal{P}_2$? $\mathcal{P}_1$ to $\mathcal{P}_3$?

Limits – Modularity



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Is it "legal" to branch $\mathcal{P}_1$ to $\mathcal{P}_2$? $\mathcal{P}_1$ to $\mathcal{P}_3$?

$\longrightarrow$ $\mathcal{P}_1 \cup \mathcal{P}_2$ is a QBN but $\mathcal{P}_1 \cup \mathcal{P}_3$ is *not* (cycle $B \to C \to D \to A \to B$)

Idea: Inputs & Outputs are insufficient, we need a **type**
Instead of a new syntax use **proof-nets** = typed graphs from Linear Logic

Plan

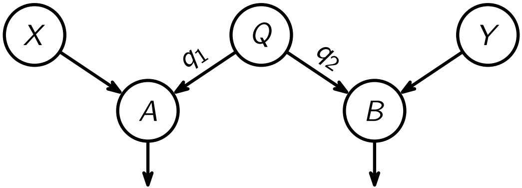
- Compositionality by Quantum Factors
- Modularity by Typing via Linear Logic

Quantum Factors

Instead of associating *quantum operations* to a node, we associate a:

Definition: Quantum Factor

Let $(X_1, \ldots, X_n)$ be random variables and $(\mathcal{H}_1, \ldots, \mathcal{H}_m)$ Hilbert spaces. A **Quantum Factor** on $(X_1, \ldots, X_n, \mathcal{H}_1, \ldots, \mathcal{H}_m)$ is a function ϕ from $\prod_{i=1}^{n} Val(X_i)$ to *positive* matrices in $\bigotimes_{j=1}^{m} \mathcal{H}_j$.



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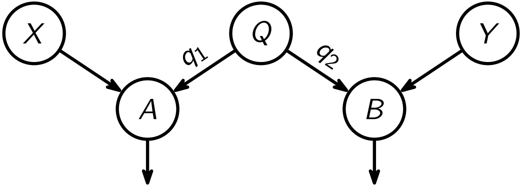
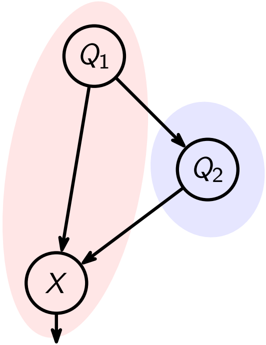
Equipped with a **product** $\odot$ and a **sum** $\sum$

If ϕ_1 is on $(\mathbb{X}_1, \mathbb{H}_1)$ and ϕ_2 on $(\mathbb{X}_2, \mathbb{H}_2)$, $\phi_1 \odot \phi_2$ is on $(\mathbb{X}_1 \cup \mathbb{X}_2, \mathbb{H}_1 \Delta \mathbb{H}_2)$

Only quantum: get $\otimes$-networks with $\odot$ = their product, $\sum$ = partial trace
Only classical: get factors of Bayesian networks with $\odot$ = their product, $\sum$ = their sum



Compositionality by Quantum Factors



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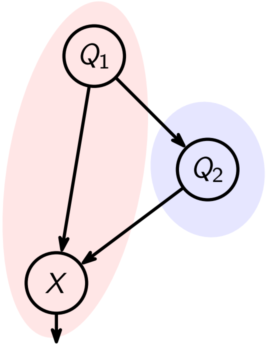

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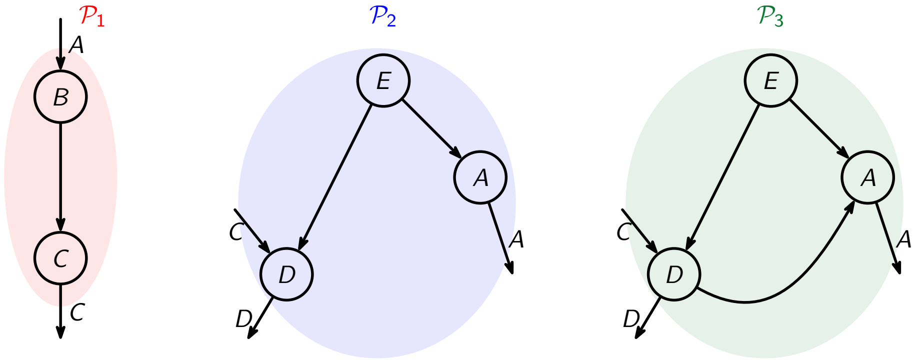
$\phi^{Q_1,X} : Val(X) \to \mathcal{H}_3 \otimes \mathcal{H}_2$

$\implies \phi^{Q_1,Q_2,X} = \phi^{Q_1,X} \odot \phi^{Q_2}$

$\phi^{Q_2} : \mathcal{H}_2 \otimes \mathcal{H}_3$

More generally, can compute for **any order** of the nodes!

Modularity



<u>Observation:</u> $\mathcal{P}_1 \cup \mathcal{P}_2$ is a QBN but not $\mathcal{P}_1 \cup \mathcal{P}_3$

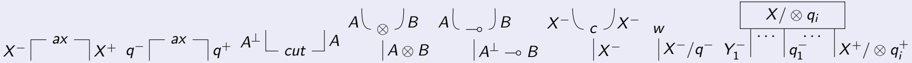
How to ensure two parts always form a QBN?

<u>Idea:</u> We add a **type** = an interface
Could do so by patching QBN and getting yet another new syntax. . .
We prefer to use **proof-nets** = typed graphs from Linear Logic

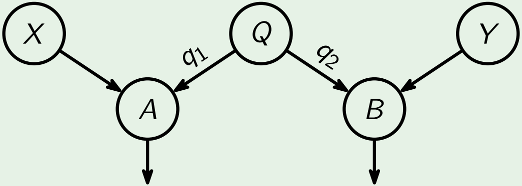
Proof-Nets for Quantum Bayesian Networks

Definition: Proof-Net

Graph respecting some **graphical criterion** and built from:



Example



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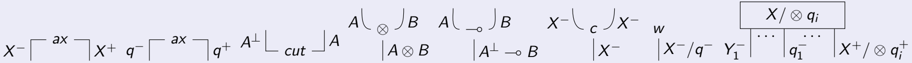

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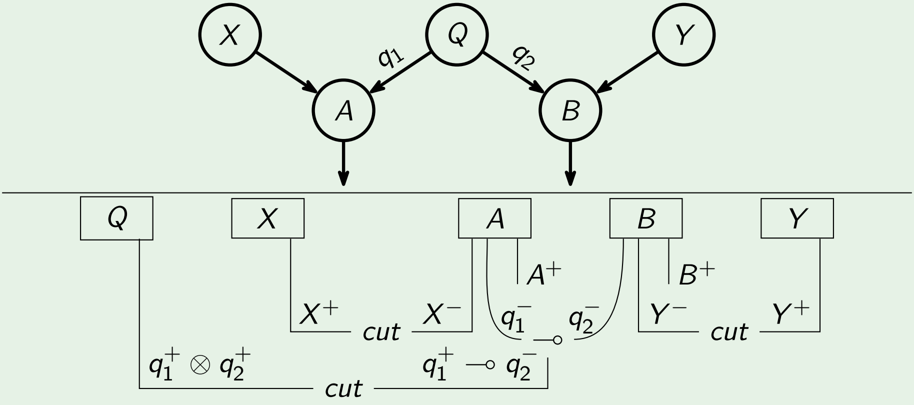
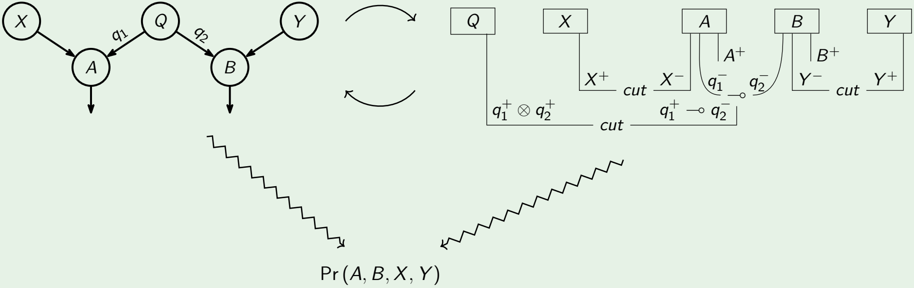


Exact Characterization by Proof-Nets

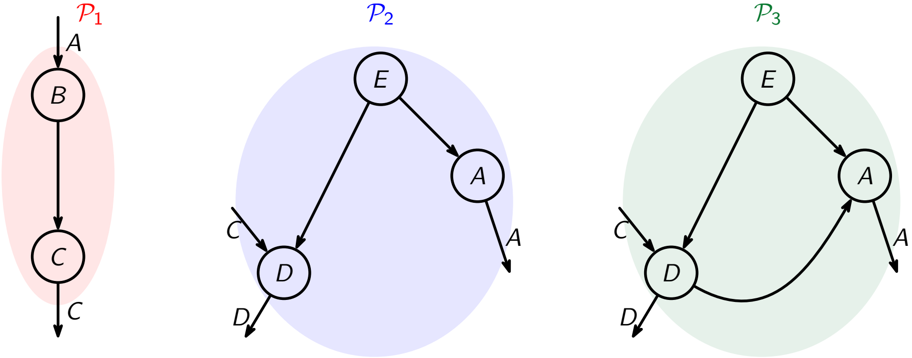
Theorem

*There is a **one-to-one correspondence** between Quantum Bayesian Networks and (a class of) Proof-Nets.*
*Furthermore, this correspondence **preserves the semantics** = the associated probability distribution.*

Example



Modularity by Typing



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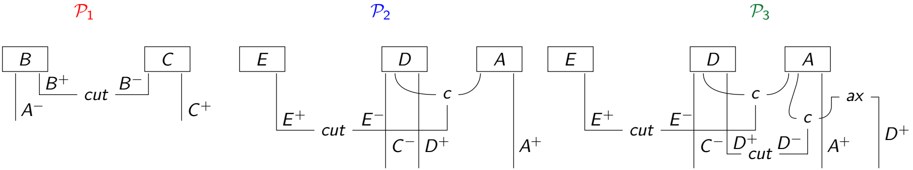

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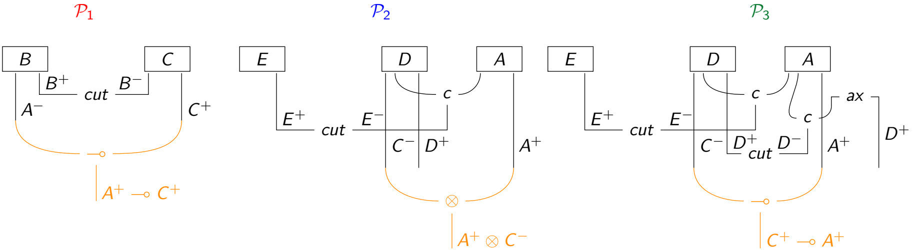

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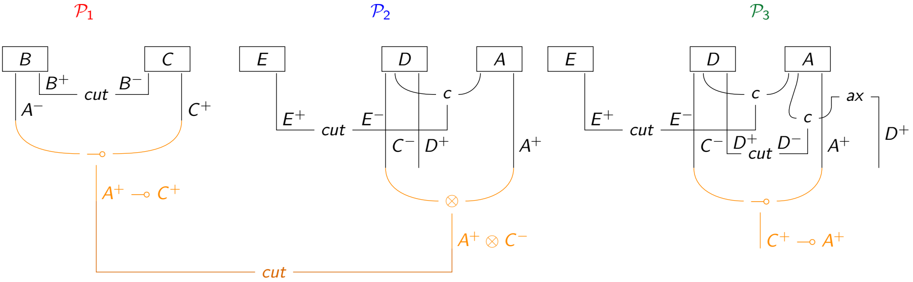

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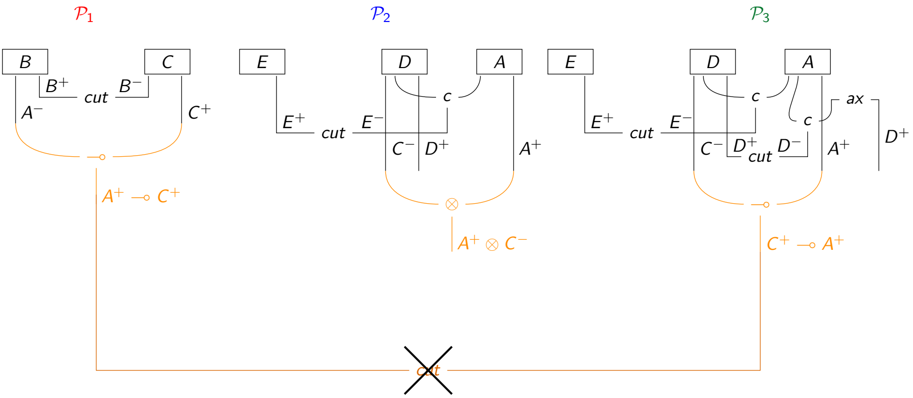

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Conclusion

Contributions

- Compositionality by modifying the semantic:
  from quantum instruments to (more general) **quantum factors**
- Conservative extension: our QBNs are **exactly** the usual Bayesian Networks in the absence of quantum nodes
- Modularity by typing in proof-nets:
  proof-theoretic approach adding an **interface**, parts with compatible interfaces are those giving a QBN

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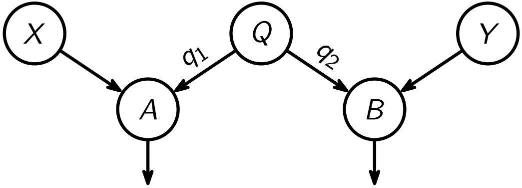
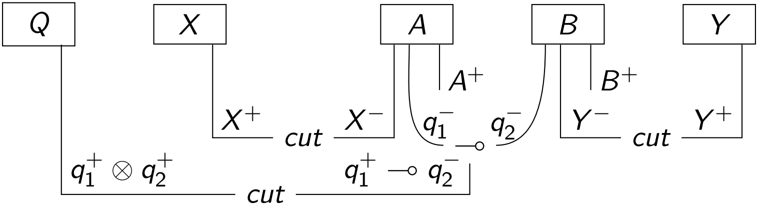
Perspectives

- Using **terms** instead of proof-nets for typing
  $\longrightarrow$ work started with Claudia Faggian and Gabriele Tedeschi
- Compositionality could be used to study **conditional independence (no-signaling)** between random variables or quantum causes



Thank you for your attention!



References

Henson, Joe, Raymond Lal, and Matthew F Pusey (Nov. 2014). "Theory-independent limits on correlations from generalized Bayesian networks". In: *New Journal of Physics* 16.11. ISSN: 1367-2630. DOI: 10.1088/1367-2630/16/11/113043.