

Cut-Expansion in Proof-Nets of Multiplicative Linear Logic

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Introduction

Cut-reduction

$$\frac{\frac{\pi_1}{\vdash B^\perp, A^\perp, \Gamma} \quad (\wp) \quad \frac{\frac{\pi_2}{\vdash A, \Delta} \quad \frac{\pi_3}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma} \quad (\otimes)}{\vdash \Gamma, \Delta, \Sigma} \quad (cut) \quad \longrightarrow \quad \frac{\frac{\pi_1}{\vdash B^\perp, A^\perp, \Gamma} \quad \frac{\pi_3}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma} \quad (cut) \quad \frac{\pi_2}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma} \quad (cut)$$

Very useful

Example of result: can reduce to a cut-free proof (weak normalization)

Applications:

- No proof of falsity \perp or 0
- Proof search
- ...

Introduction

Cut-expansion

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Useful?

Example of result: can expand to a proof with only one cut-rule, at its root

Applications:

- Craig's interpolation – following [Sau25]¹ and [FOS25]²
- Denotational semantic – used in [EFP24]³

¹ Interpolation as Cut-Introduction: On the Computational Content of Craig-Lyndon Interpolation

² On Correctness, Sequentialization and Interpolation

³ Bayesian Networks and Proof-Nets: a proof-theoretical account of Bayesian Inference

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This talk: Simple proof of this result in *proof-nets*, adaptable enough to be transferred in most fragments of LL

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- ▶ Proof-Nets & Cut-Reduction
- ▶ Expanding to a unique *cut* & Applications
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
- ▶ Extension to full Linear Logic

Proof-nets for unit-free Multiplicative Linear Logic

- Formula

$A, B ::= X \mid X^\perp \mid A \otimes B \mid A \wp B$

- Orthogonal

$(X^\perp)^\perp = X$ $(A \otimes B)^\perp = B^\perp \wp A^\perp$
 $(A \wp B)^\perp = B^\perp \otimes A^\perp$

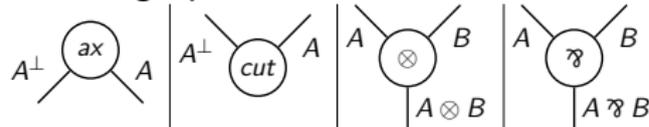
Proof-nets for unit-free Multiplicative Linear Logic

- **Formula**

$A, B ::= X \mid X^\perp \mid A \otimes B \mid A \wp B$

- **Proof-Structure**

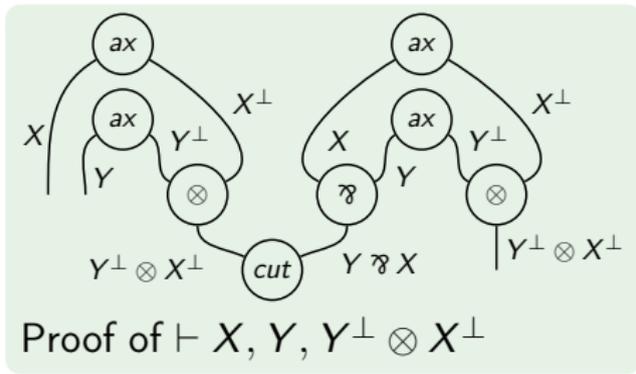
Partial graph built from:



Pending edges = sequent proved

- **Orthogonal**

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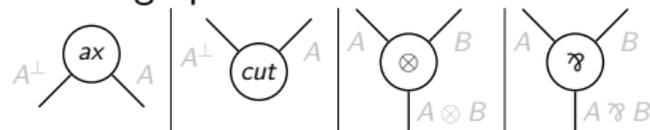
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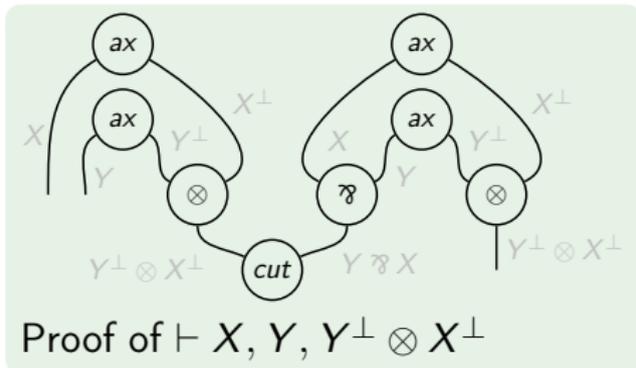


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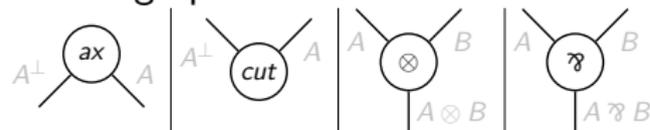
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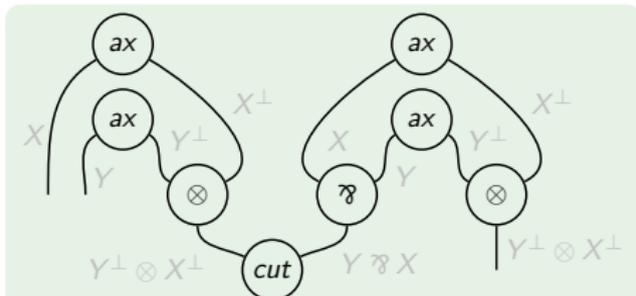
- **Proof-Net**

Danos-Regnier correctness criterion:
each cycle contains the two edges above some \wp

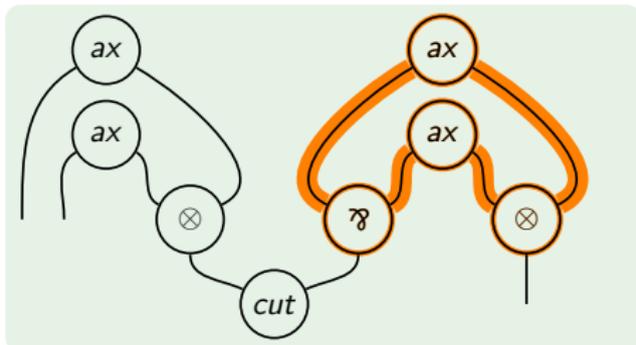
We work with *mix* to simplify, but everything also holds without it.

- **Orthogonal**

$(X^\perp)^\perp = X \quad (A \otimes B)^\perp = B^\perp \wp A^\perp$
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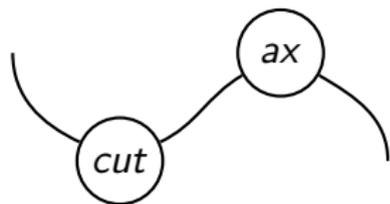


Proof of $\vdash X, Y, Y^\perp \otimes X^\perp$

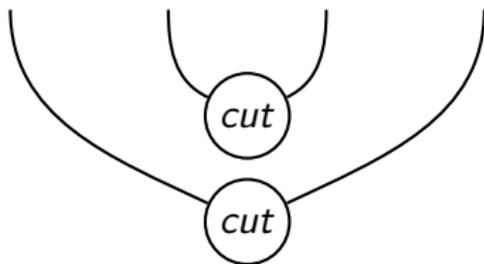
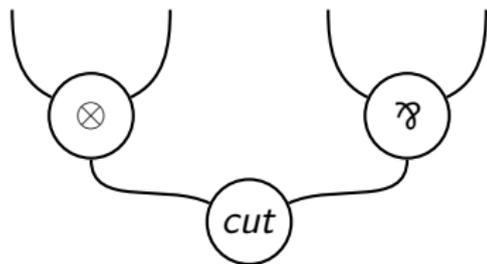


Cut-Reduction

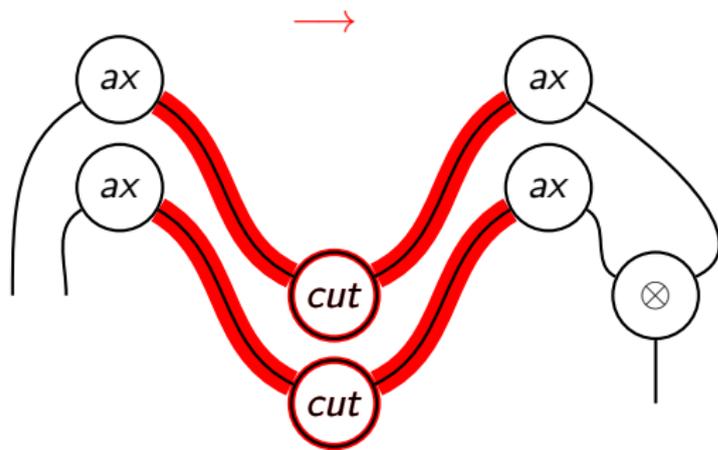
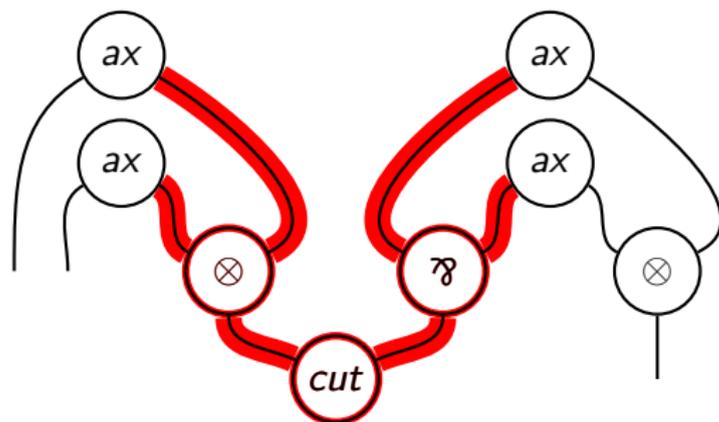
- *ax* case



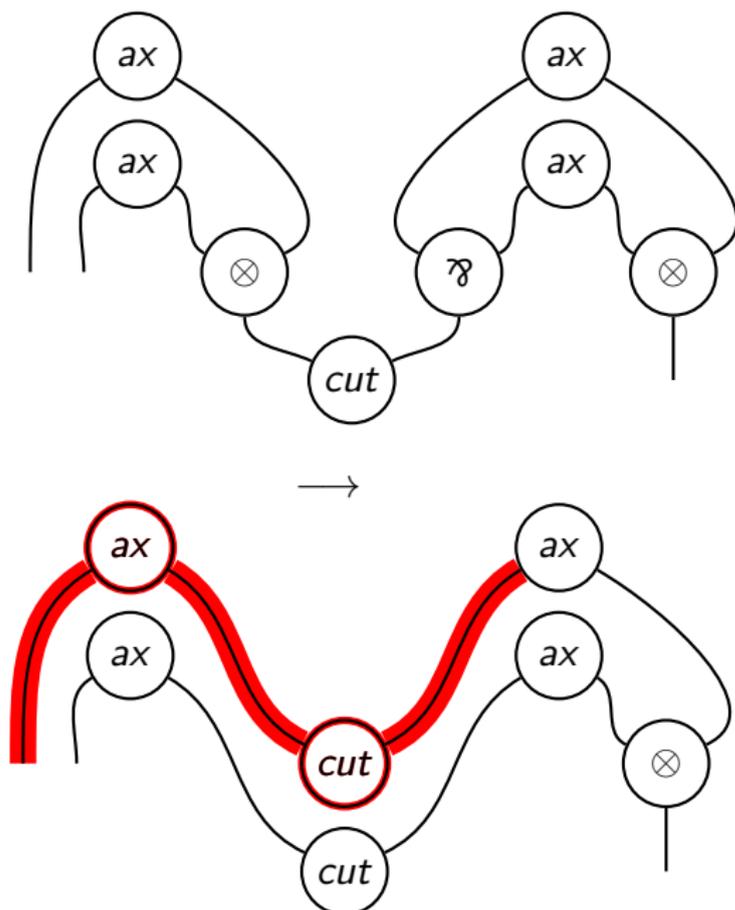
- $\wp - \otimes$ case



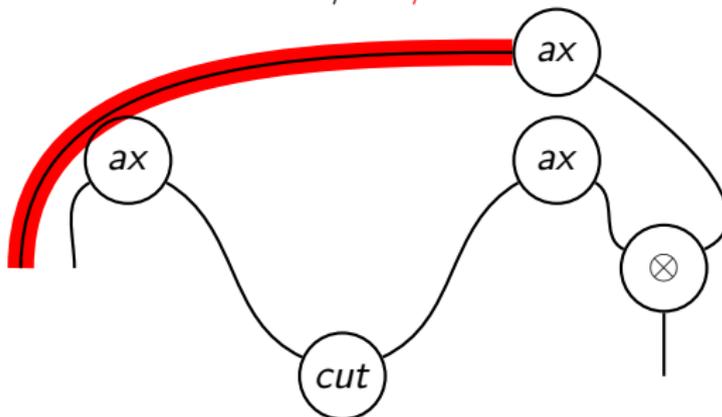
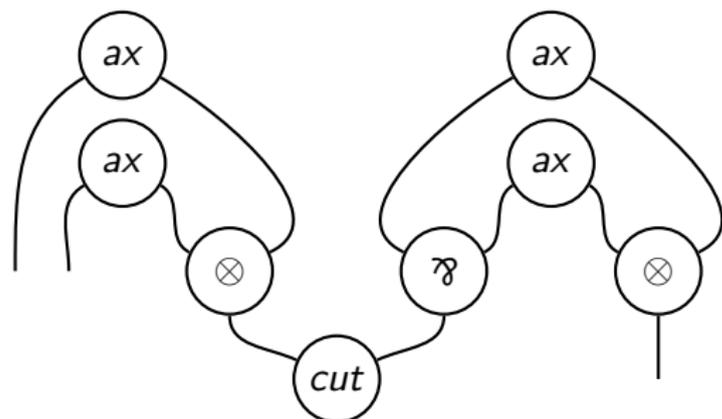
Cut-reduction on an example



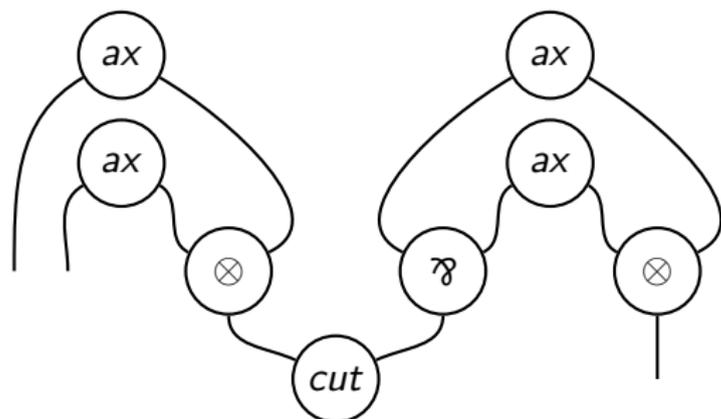
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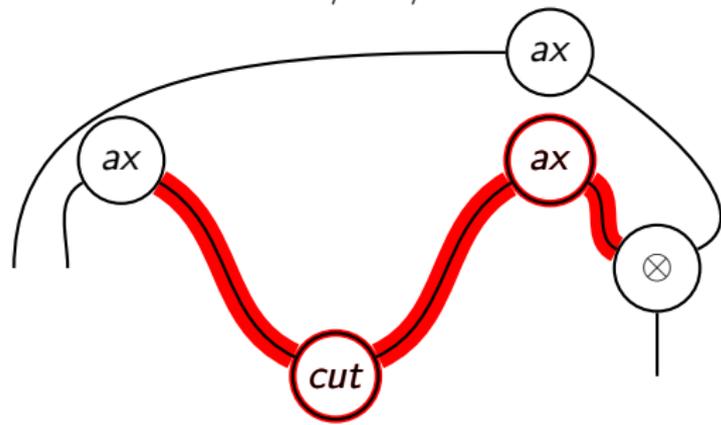
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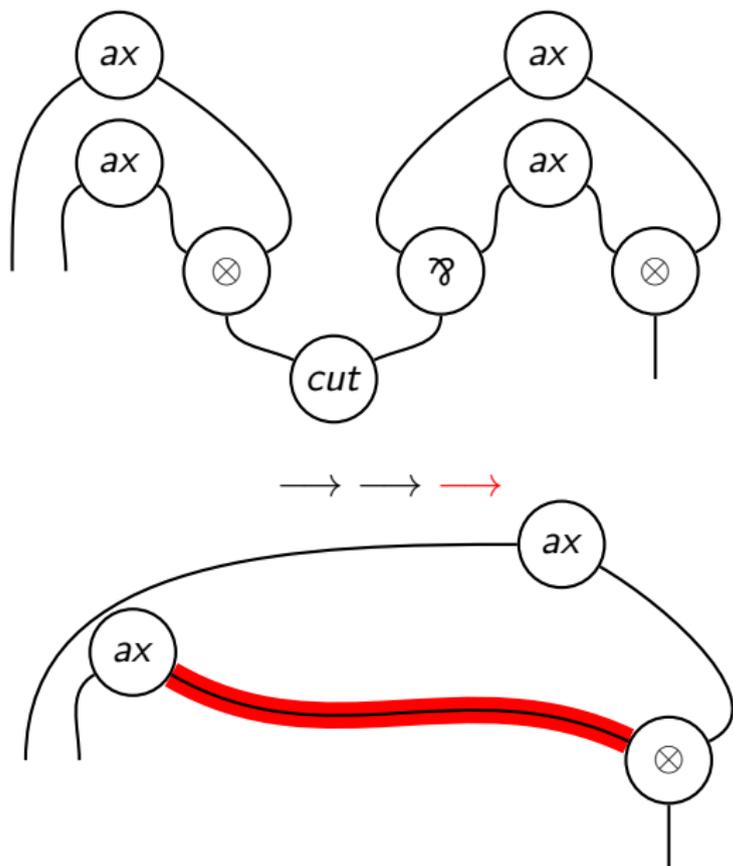
Cut-reduction on an example



→ →



Cut-reduction on an example



On Correctness and Cut-expansion

Has a cut? Can always reduce it.

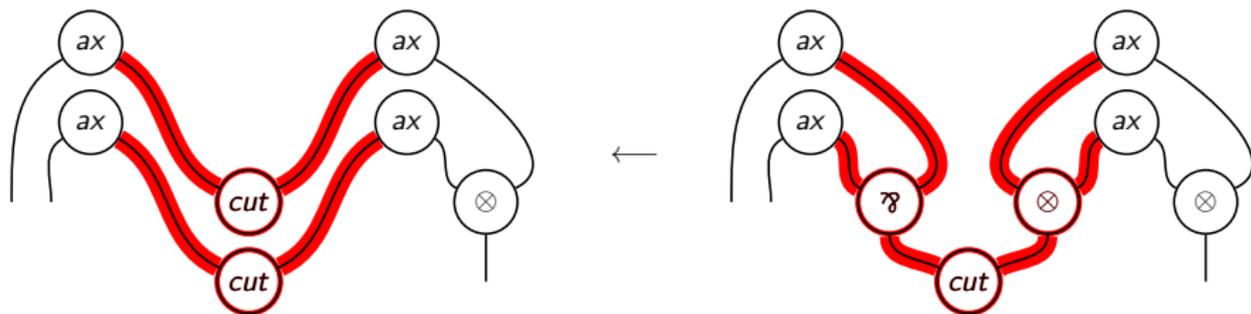
Lemma

If \mathcal{P} is a proof-net and $\mathcal{P} \rightarrow \mathcal{Q}$, then \mathcal{Q} is a proof-net.

Has 2 cuts? Can always expand them.

Fact

If \mathcal{P} is a proof-net and $\mathcal{P} \leftarrow \mathcal{Q}$, then \mathcal{Q} **may not be** a proof-net.



On Correctness and Cut-expansion

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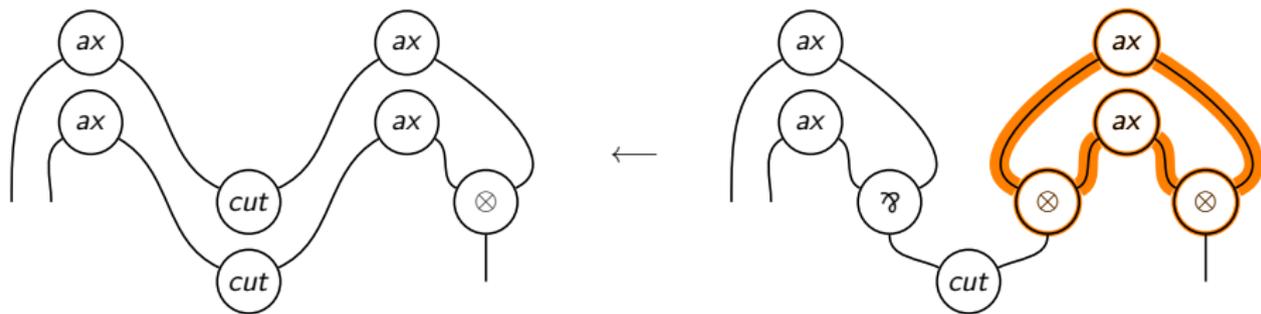
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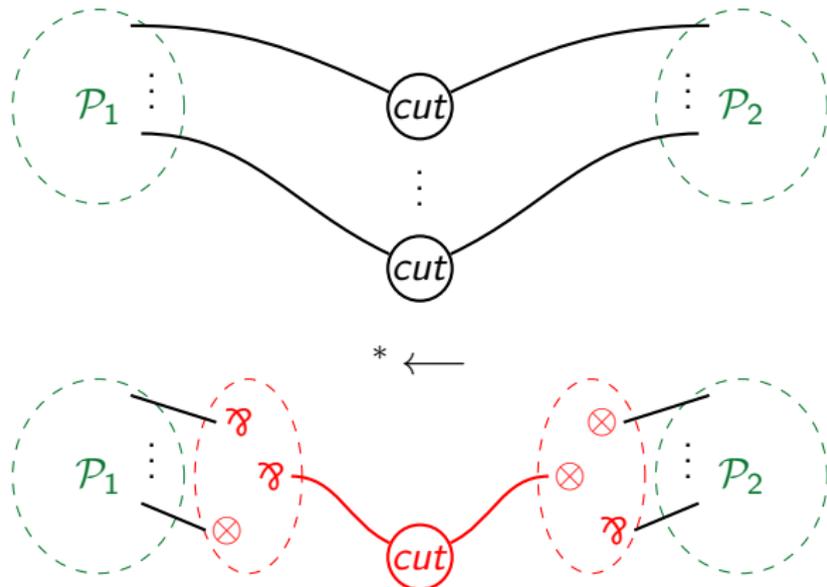


- ▶ Proof-Nets & Cut-Reduction
- ▶ **Expanding to a unique *cut* & Applications**
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
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Expanding to a unique cut

Proposition: Expanding to a unique cut

Take a proof-net \mathcal{P} made of two sub-graphs linked by $n \geq 1$ cut-vertices. There is a sequence of $n - 1$ $\wp - \otimes$ cut-expansion steps on these vertices yielding a **proof-net**.



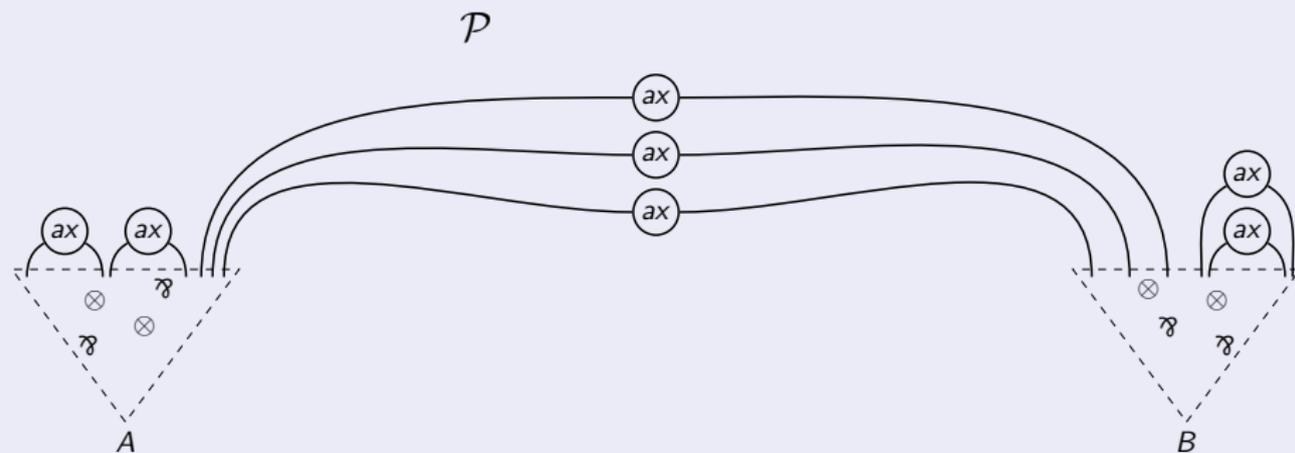
Application: Craig's Interpolation

Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P} \ast \leftarrow \mathcal{P}_1 \text{ cut } \mathcal{P}_2$.

New Proof.



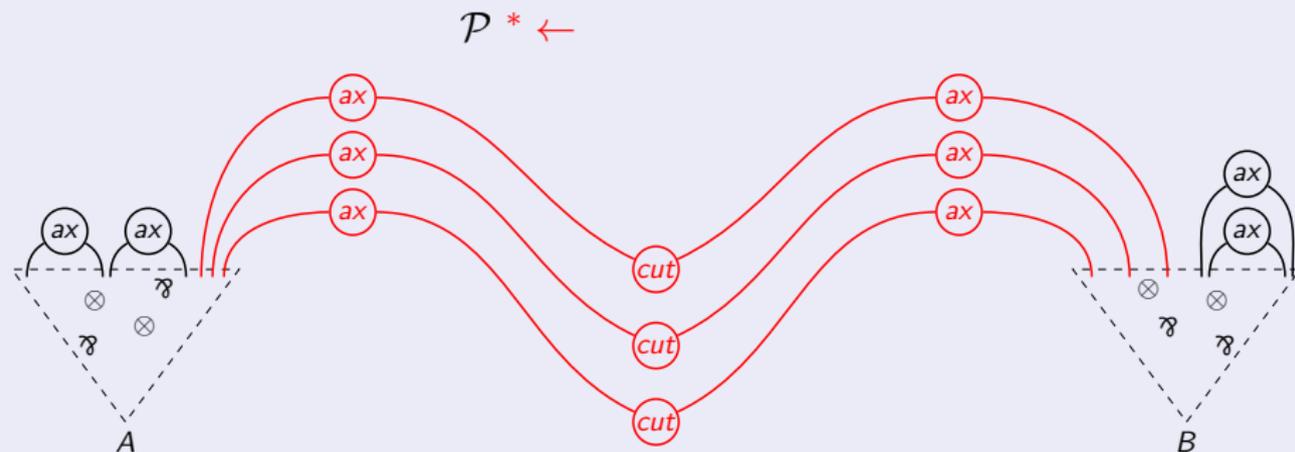
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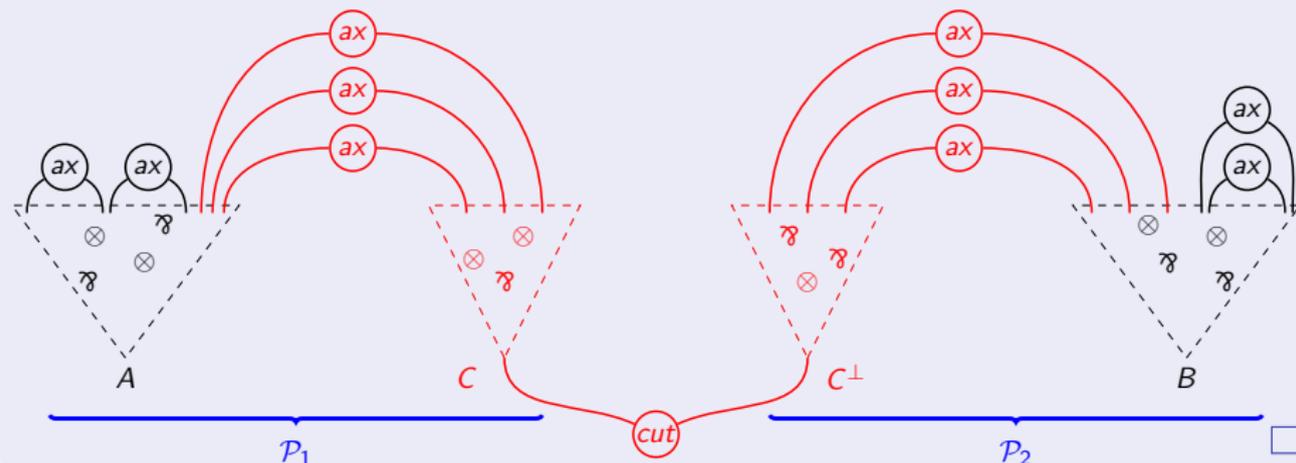
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New Proof.

$$\mathcal{P} \ast \leftarrow \ast \leftarrow \mathcal{P}_1 \text{ cut } \mathcal{P}_2$$



Differences with [FOS25]

Proof-relevant Craig's interpolation [Sau25; FOS25]

Set \mathcal{P} a *connected* cut-free proof-net of $\vdash A, B$.

There is a formula C whose atoms appear in both A and B and cut-free proof-nets \mathcal{P}_1 of $\vdash A, C$ and \mathcal{P}_2 of $\vdash C^\perp, B$ such that $\mathcal{P} \ast \leftarrow \mathcal{P}_1$ cut \mathcal{P}_2 .

No mix-rule \implies connected proof-net.

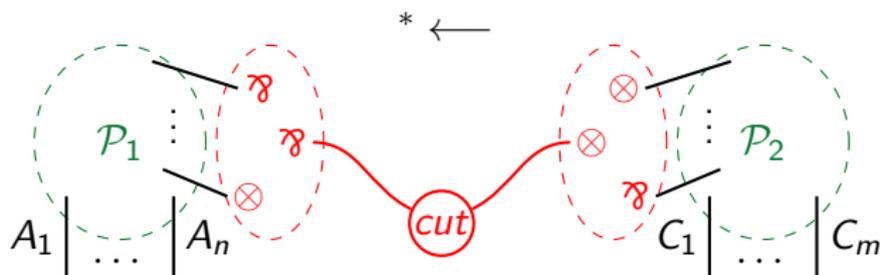
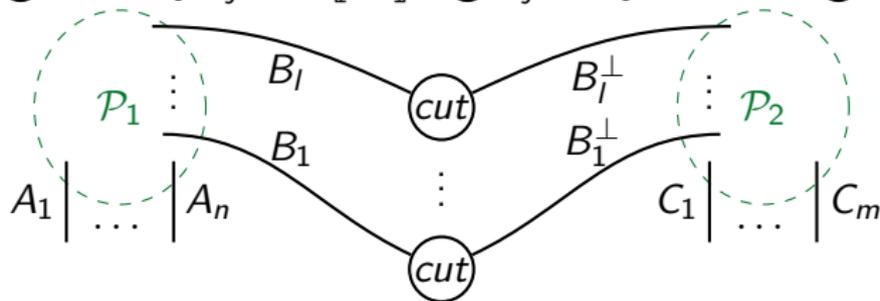
[FOS25] also proves Craig's interpolation in proof-nets but:

- it needs the **units** \perp and 1 , even if \mathcal{P} has none;
- it proceeds by **parsing** (*i.e.* goes top-down) so cannot have **mix**;
meanwhile, we will proceed by **splitting vertices** (*i.e.* bottom-up).

Application: Denotational Semantic

Useful for **compositionality** (e.g. in [EFP24]): given *interpretations* $\llbracket \mathcal{P}_1 \rrbracket$ and $\llbracket \mathcal{P}_2 \rrbracket$, how to get the *interpretation* of the full net?

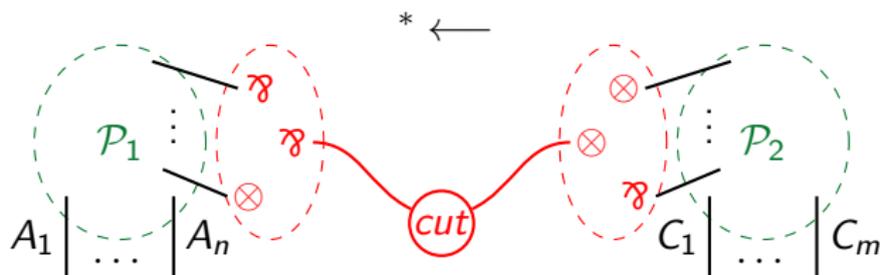
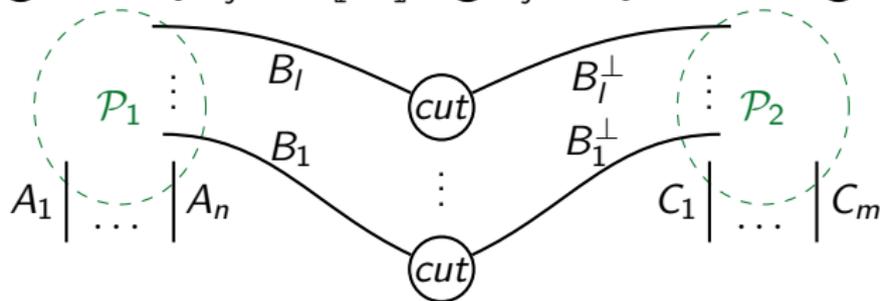
↪ Easy when **only one cut**, otherwise *typing* problem: given $\llbracket \mathcal{P}_1 \rrbracket : \otimes A_i \rightarrow \wp B_j$ and $\llbracket \mathcal{P}_2 \rrbracket : \otimes B_j \rightarrow \wp C_k$, want $\otimes A_i \rightarrow \wp C_k$



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[EFP24]'s solution with the *same result* as here, but using **parsing** again!

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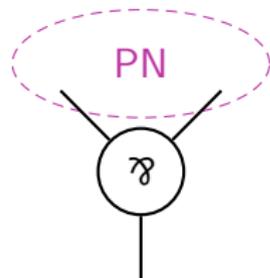
Reasoning

Proposition: Expanding to a unique *cut*

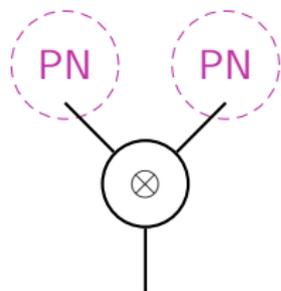
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- Proof by induction on $\#|\text{vertices}|$
- Main consideration: when adding a \otimes , we ensure it creates no “bad cycle”
- Case study using a proof-net with a *cut* always has a **splitting vertex**:

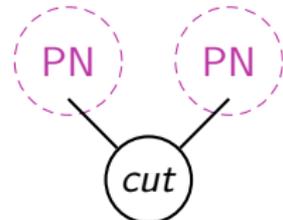
- \wp -vertex



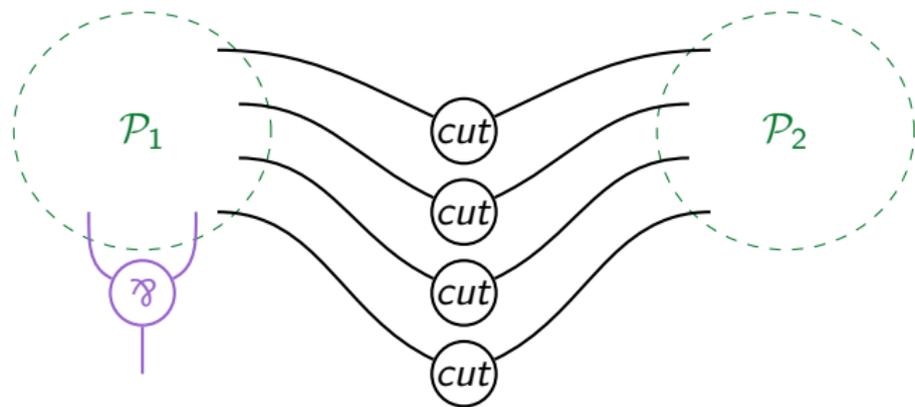
- \otimes -vertex



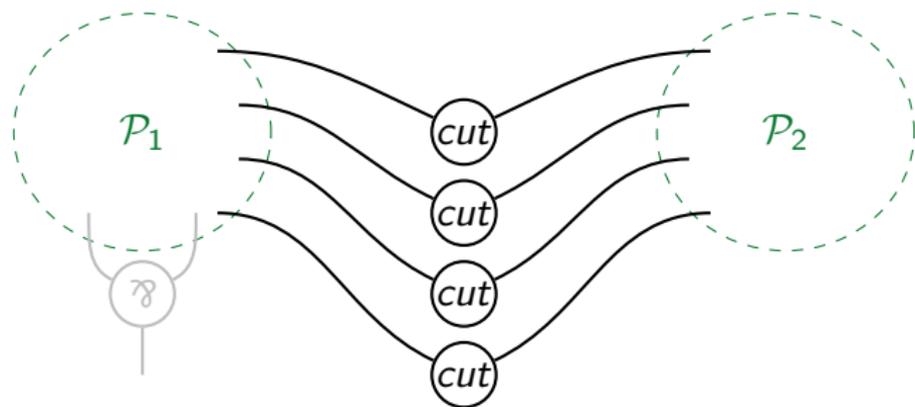
- *cut*-vertex



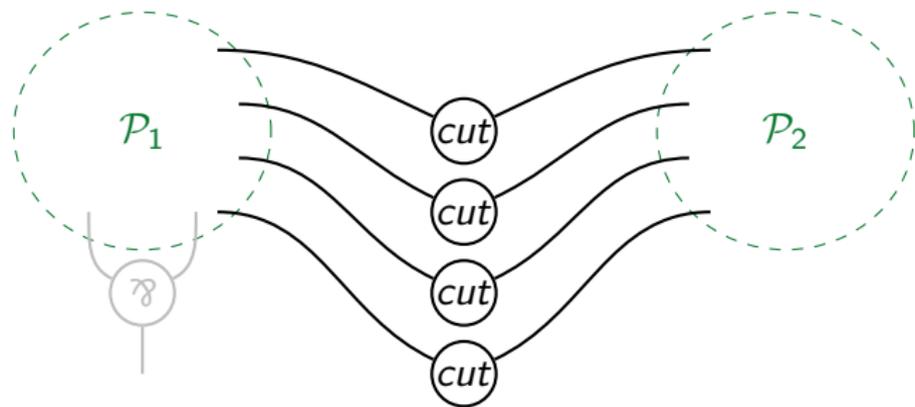
Splitting \mathcal{F}



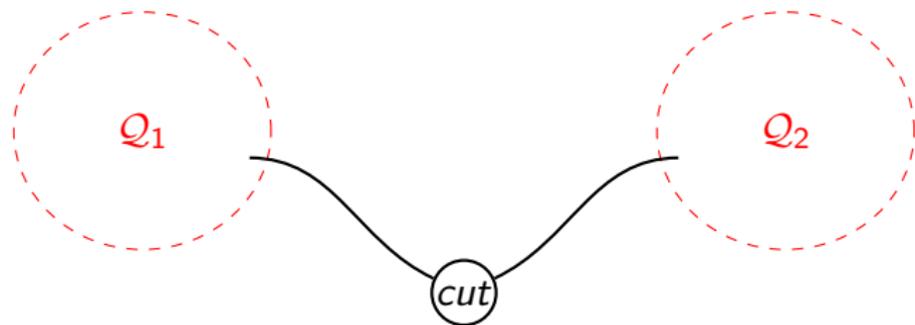
Splitting \mathcal{P}



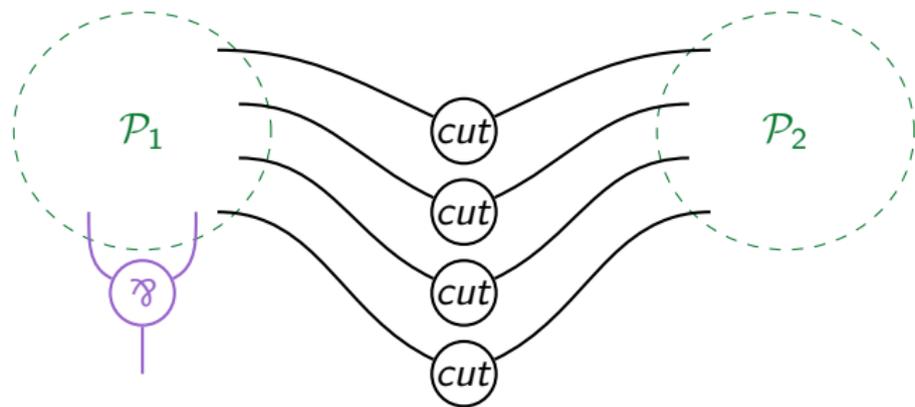
Splitting γ



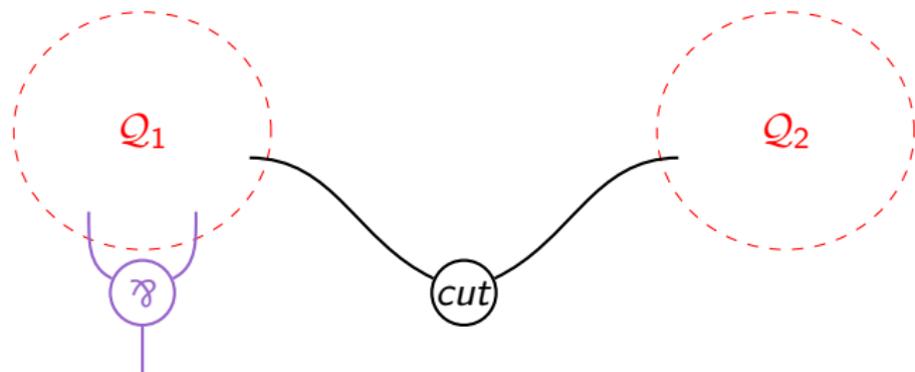
* \leftarrow



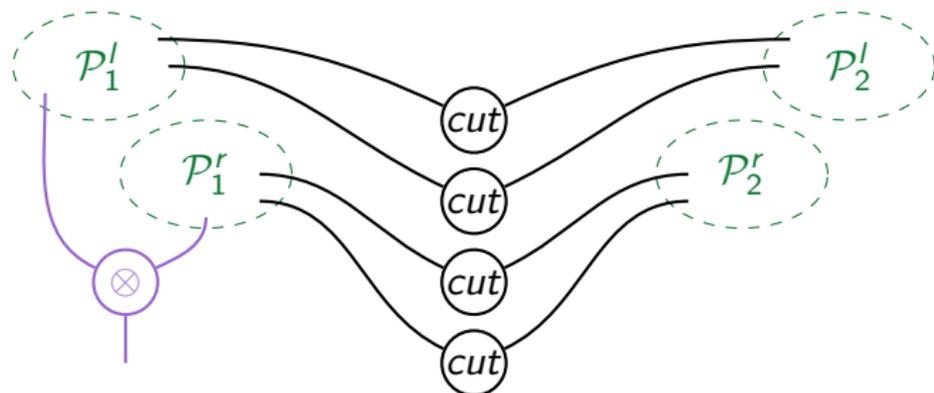
Splitting \wp



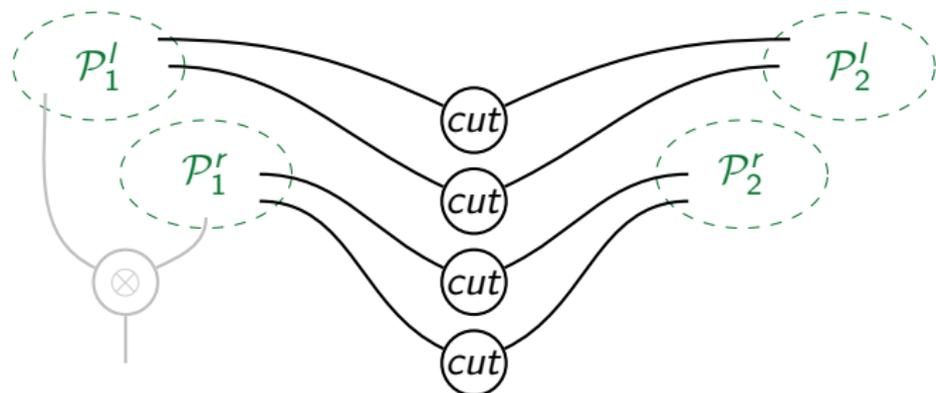
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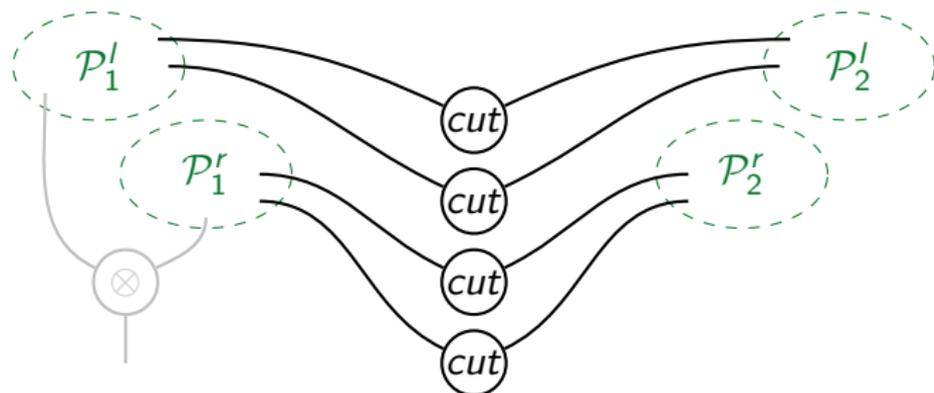
Splitting \otimes



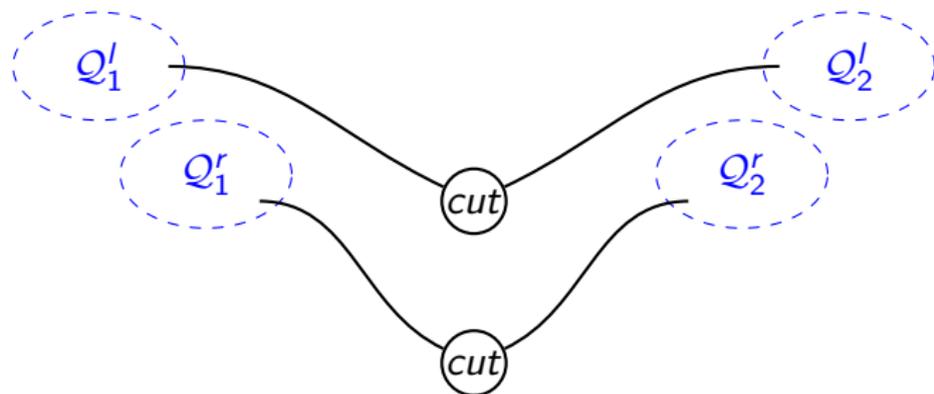
Splitting \otimes



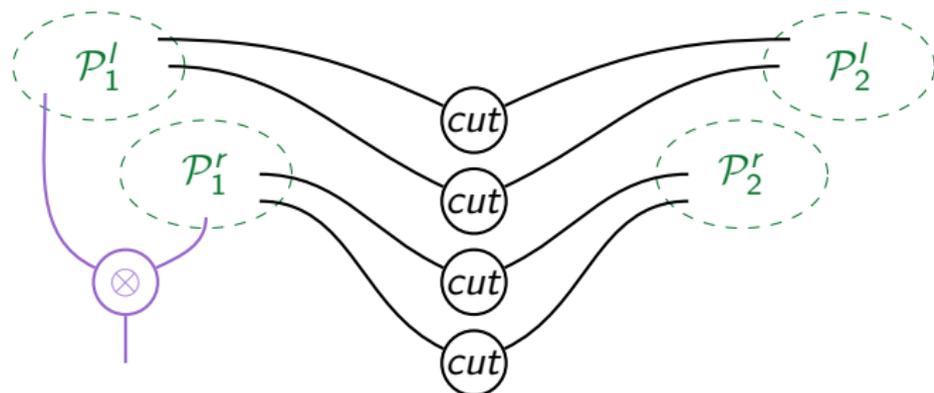
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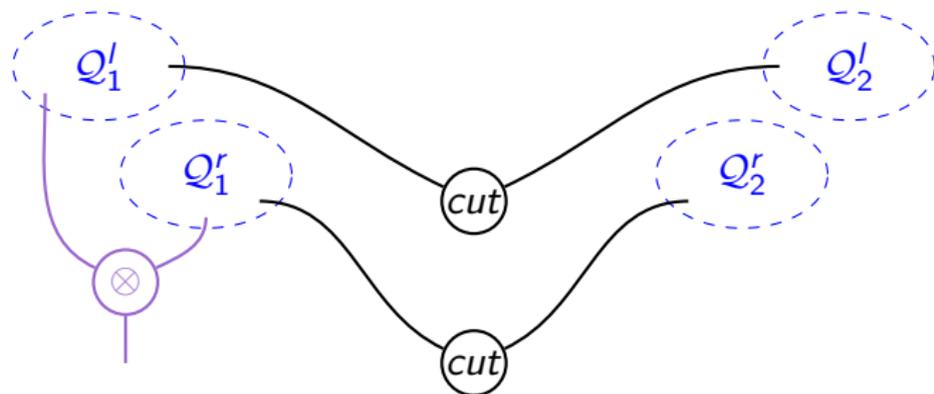
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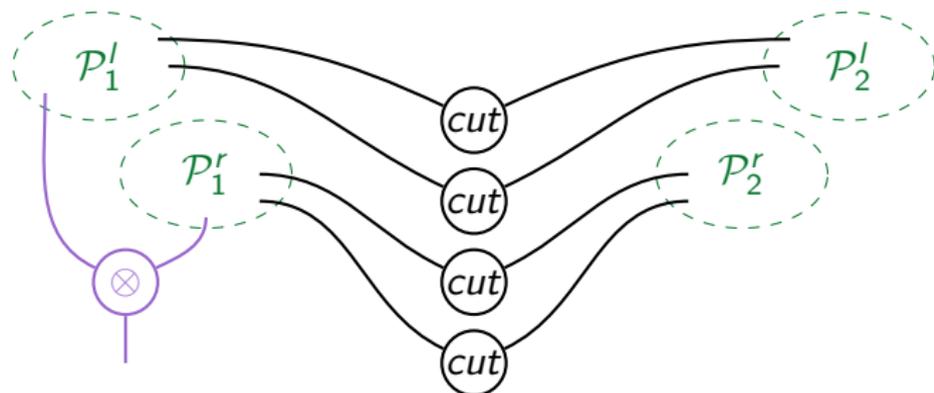
Splitting \otimes



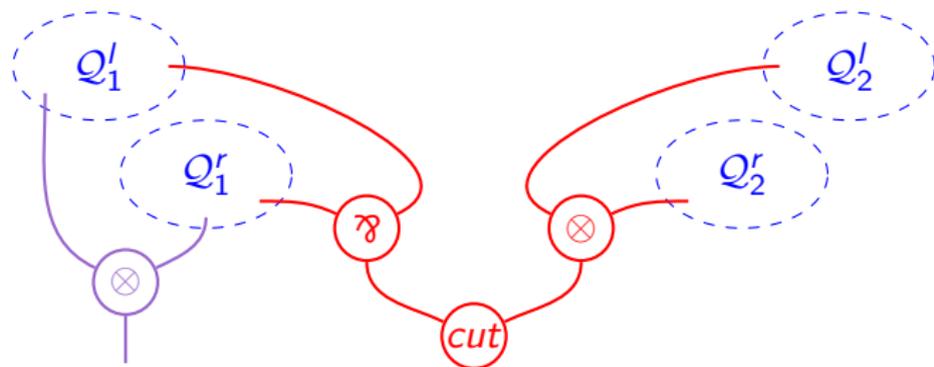
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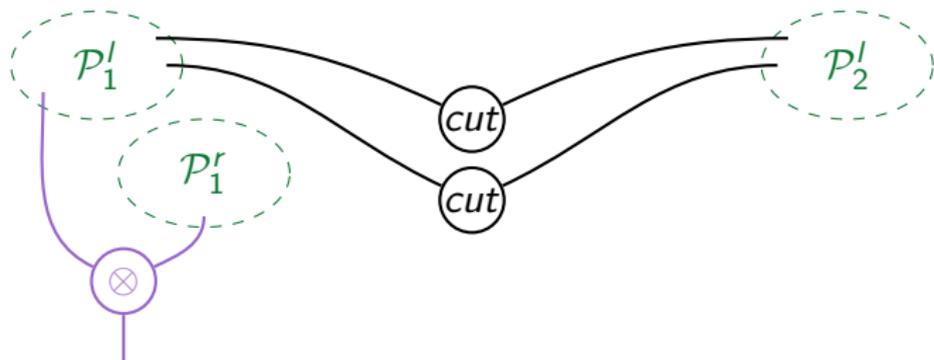
Splitting \otimes



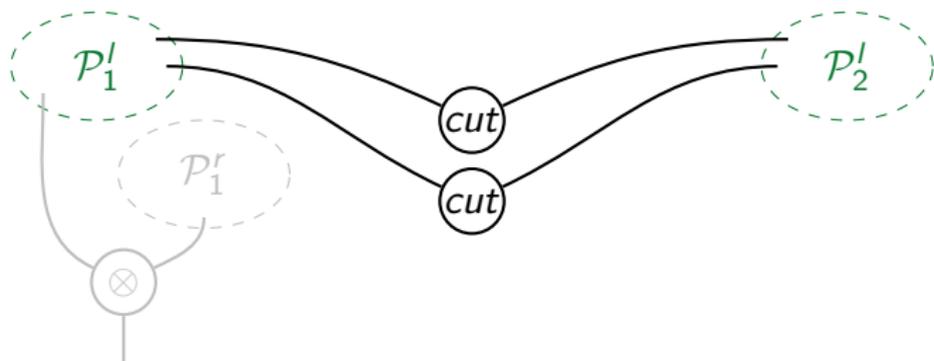
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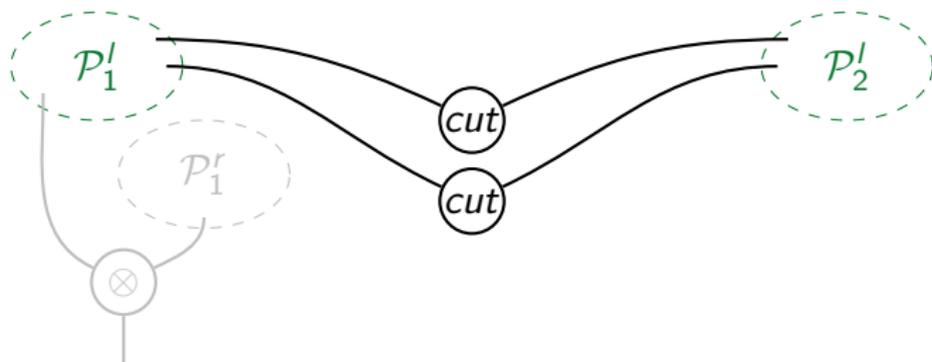
Splitting \otimes *bis*



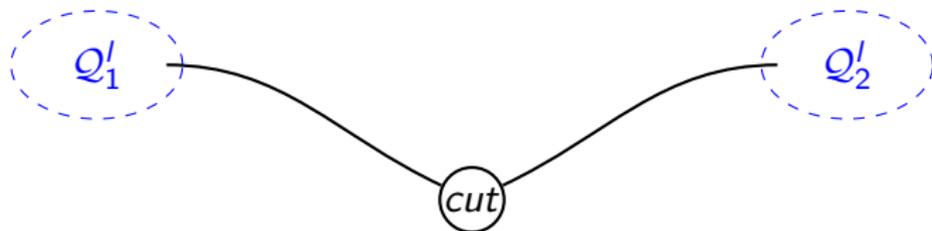
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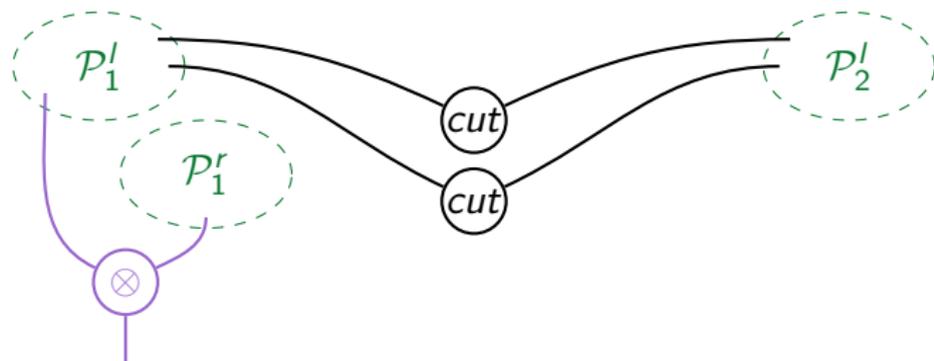
Splitting \otimes bis



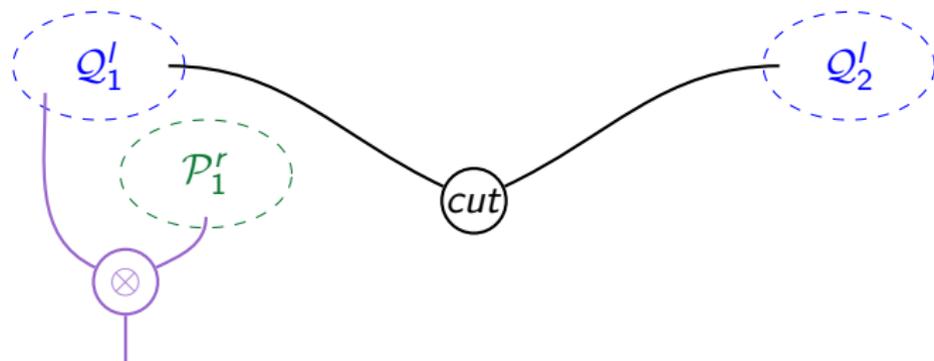
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Splitting \otimes bis

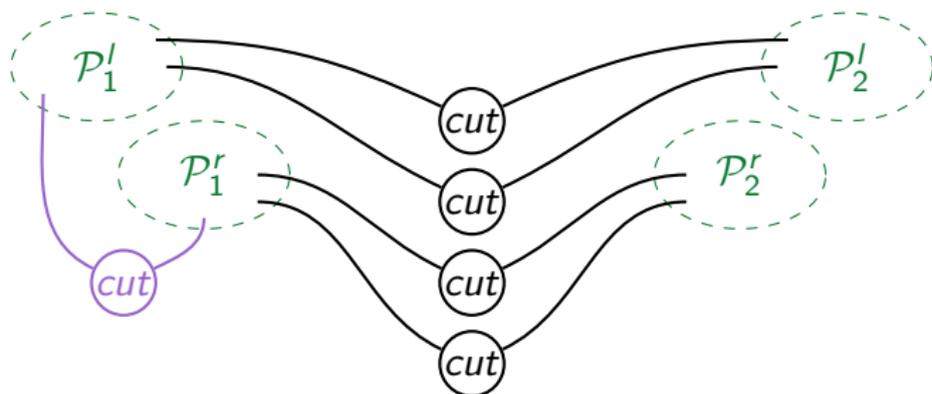


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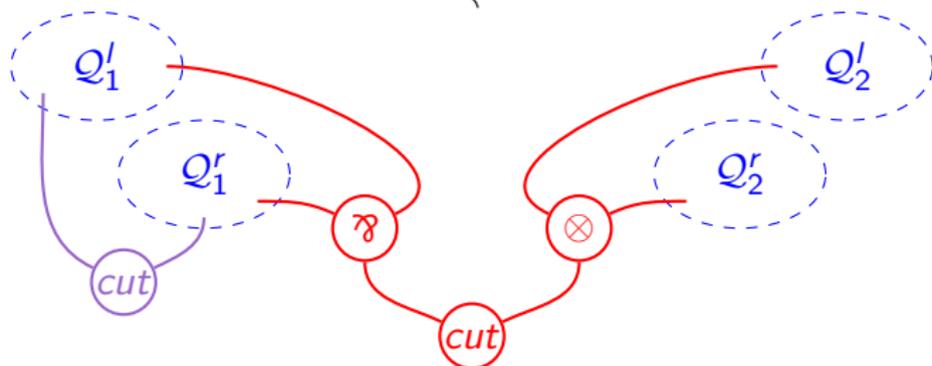


Splitting *cut*

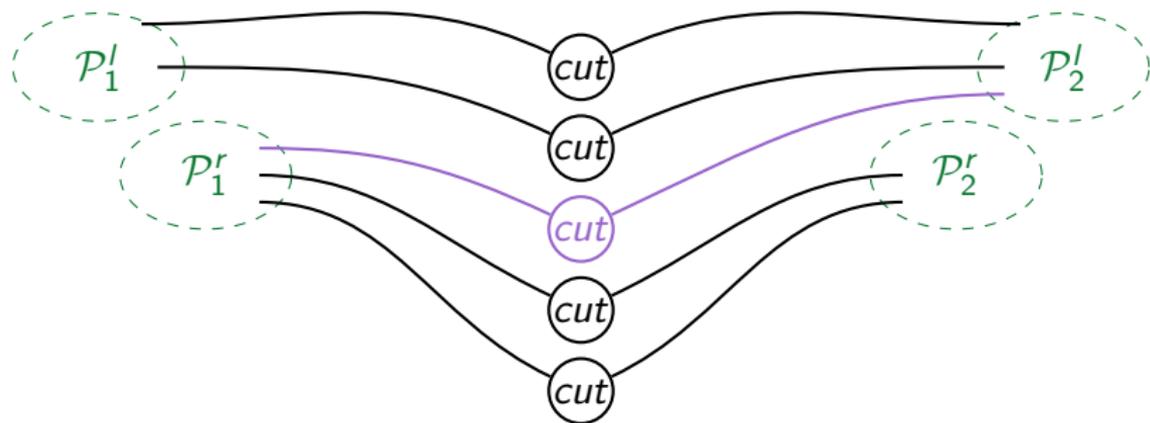
If this *cut* is not between the two sub-graphs: same as the \otimes case.



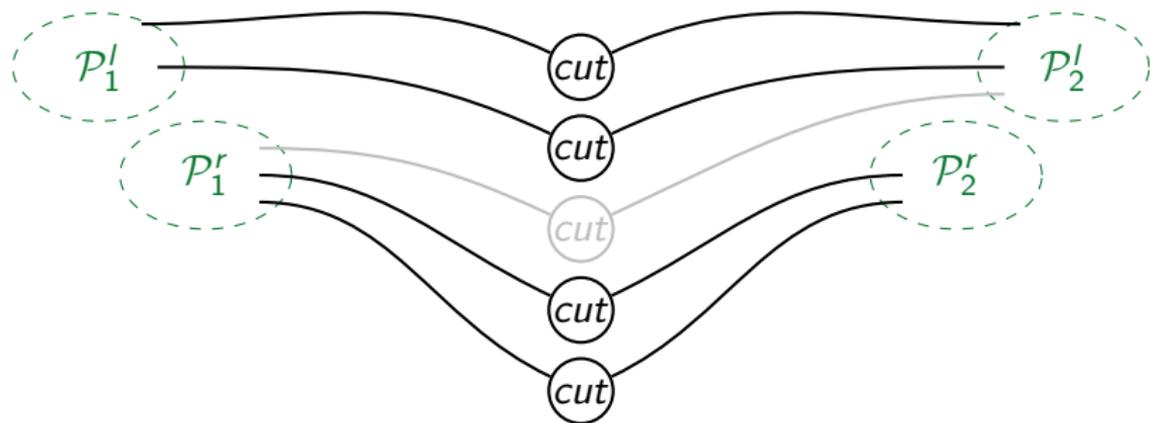
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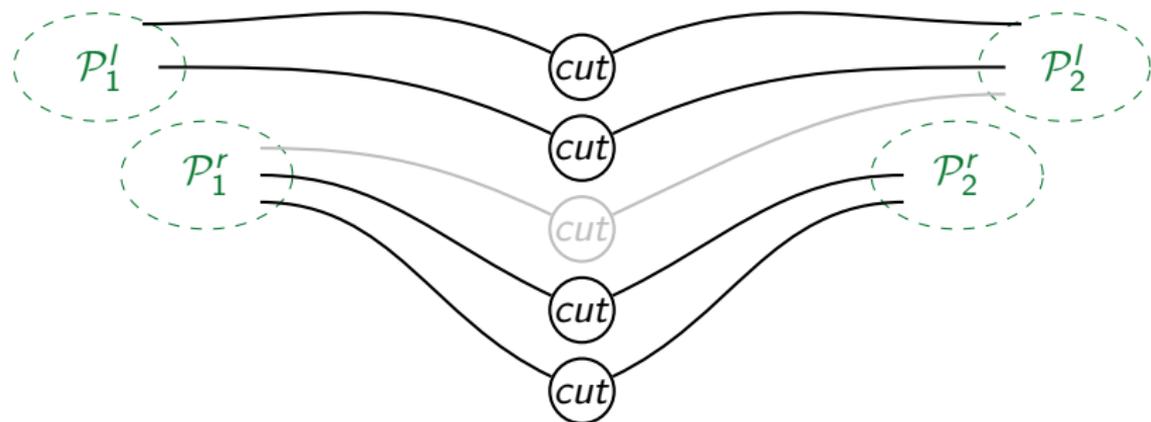
Splitting *cut*



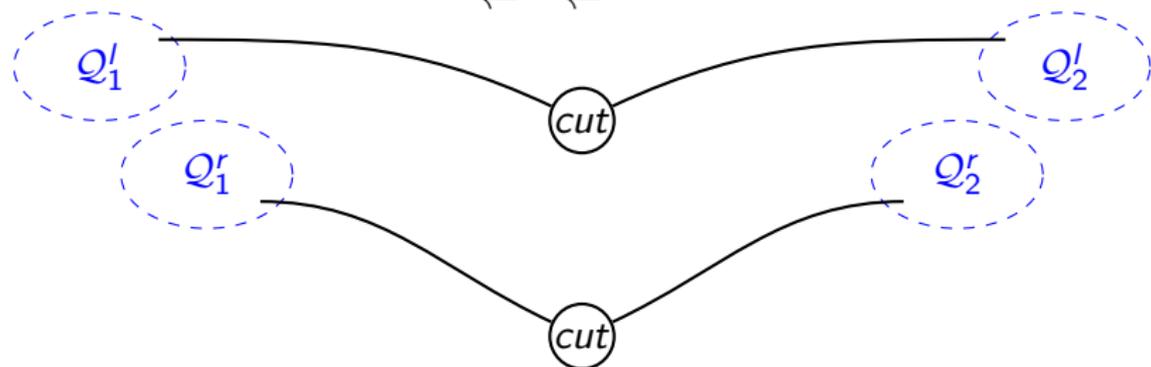
Splitting *cut*



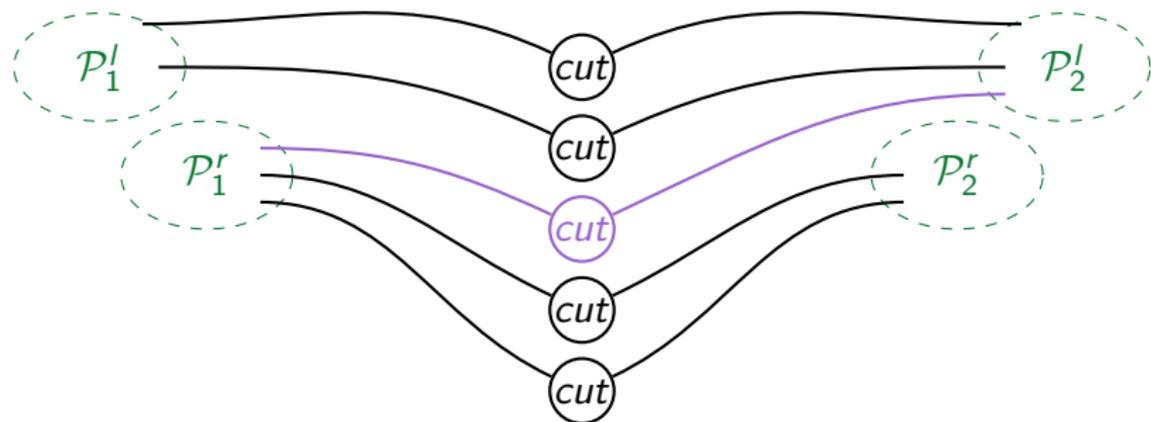
Splitting *cut*



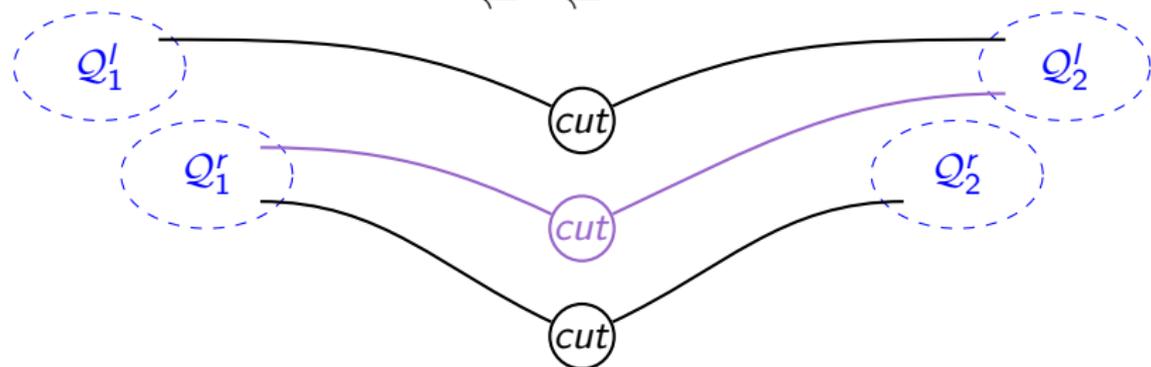
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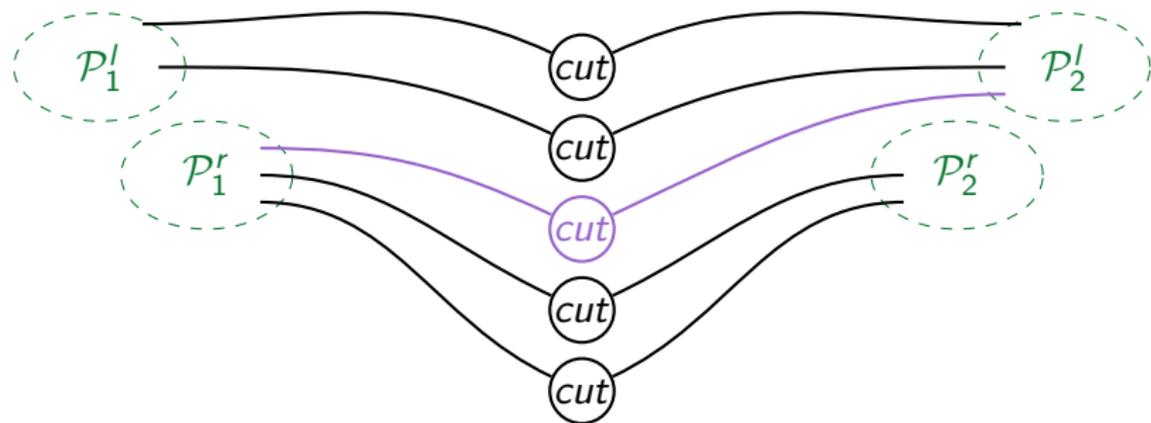
Splitting *cut*



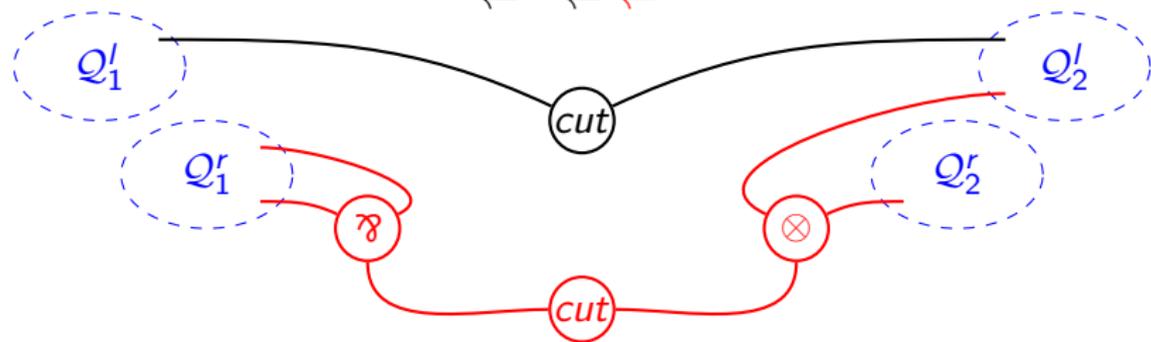
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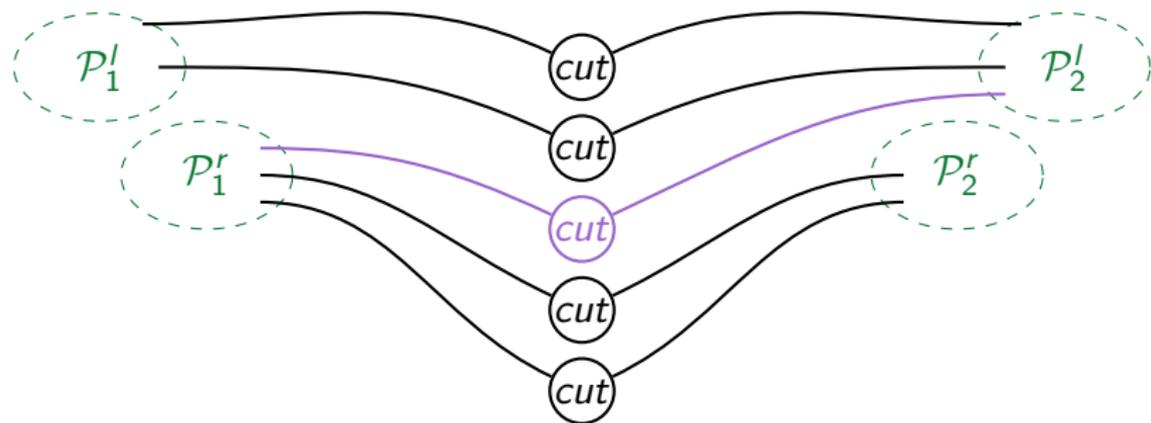
Splitting *cut*



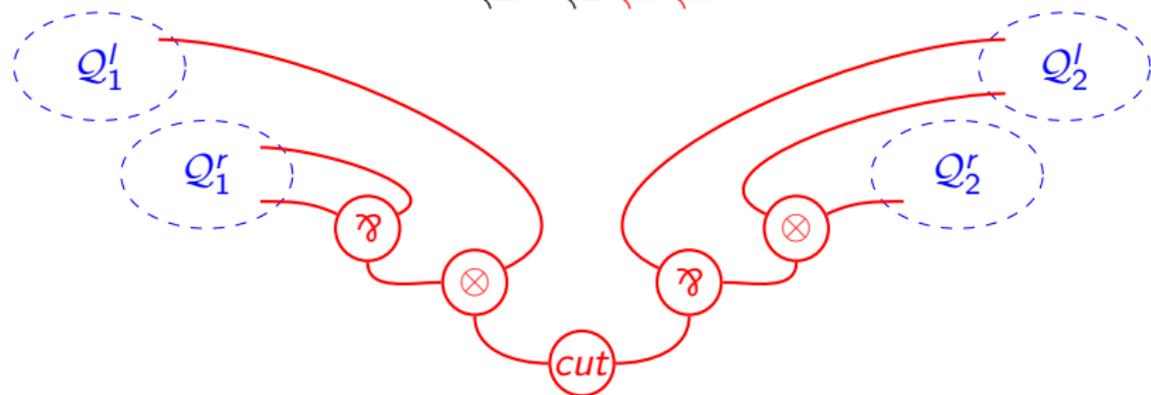
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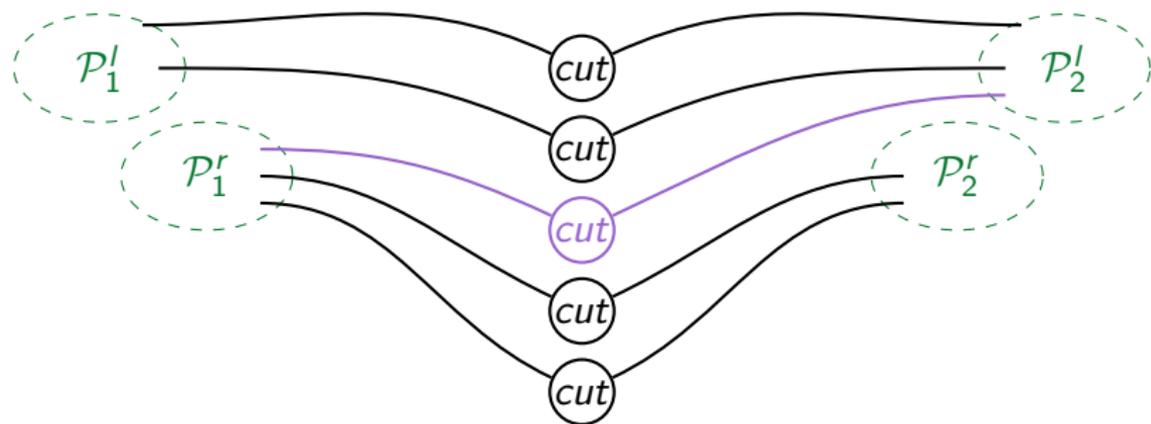
Splitting *cut*



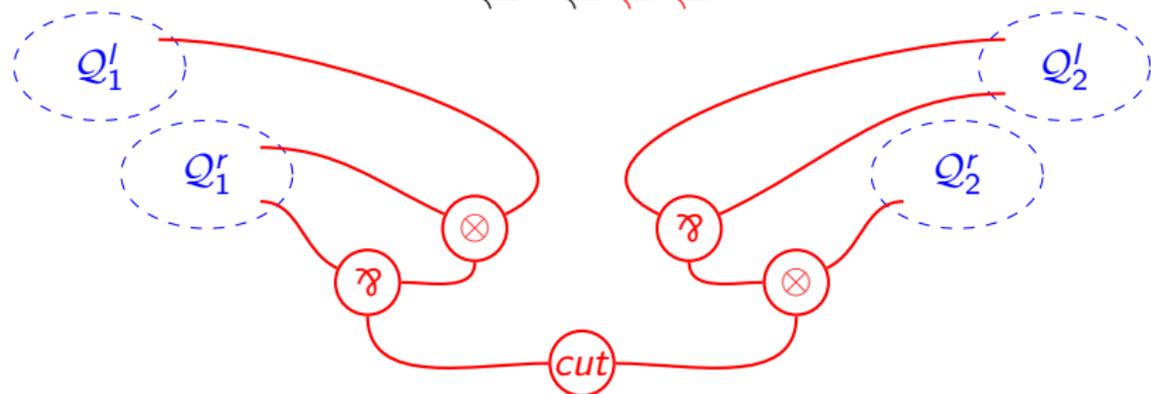
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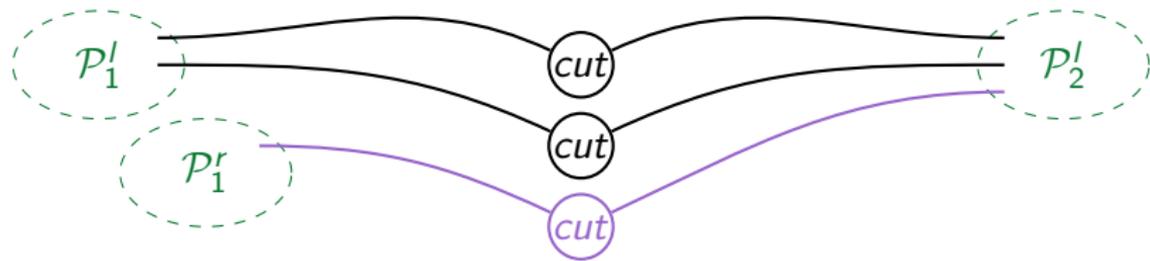
Splitting *cut*



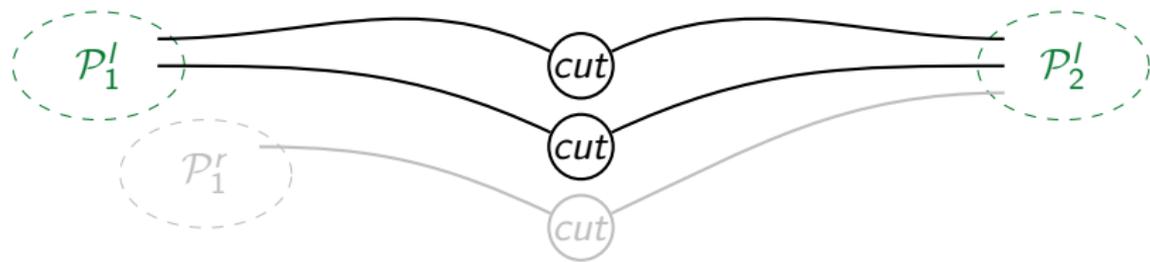
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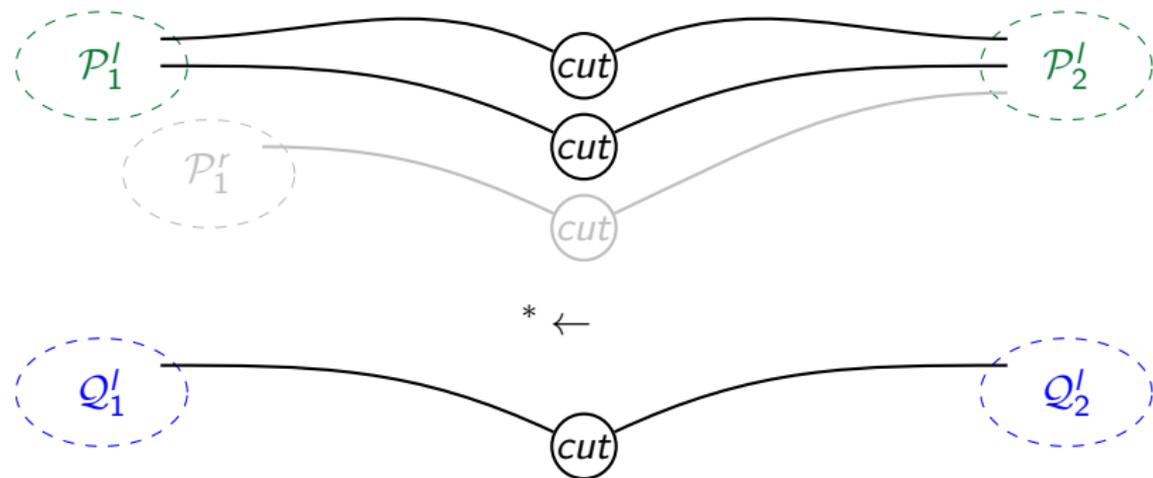
Splitting *cut bis*



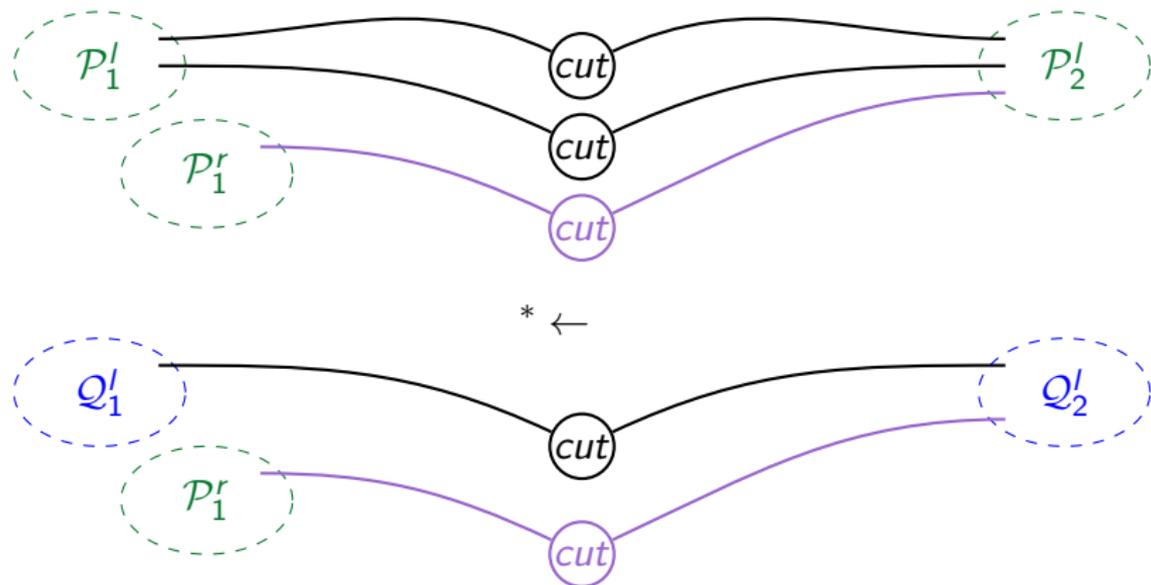
Splitting *cut bis*



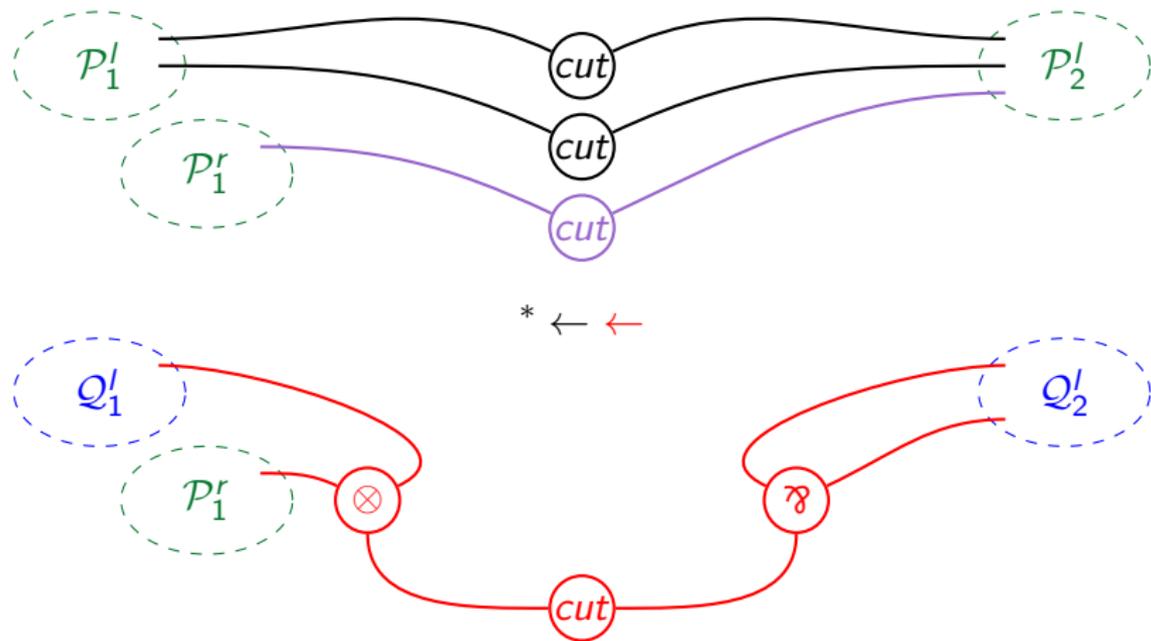
Splitting *cut bis*



Splitting *cut bis*



Splitting *cut bis*

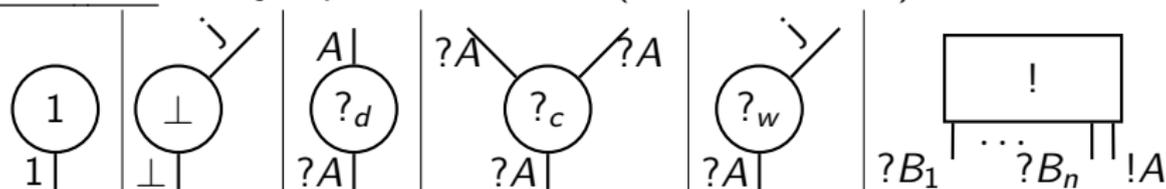


- ▶ Proof-Nets & Cut-Reduction
- ▶ Expanding to a unique *cut* & Applications
 - Craig's Interpolation
 - Denotational Semantic
- ▶ Proof of the Expansion to a unique *cut*
- ▶ Extension to full Linear Logic

In Proof-Nets for MLL with units & MELL

Same as for unit-free MLL!

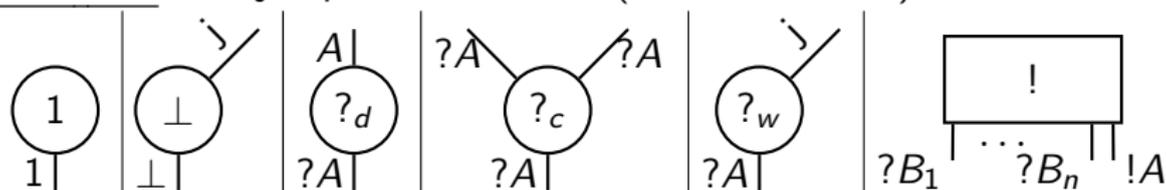
- Proof-nets with jumps for \perp and $?_w$ (or the mix-rules) and boxes for $!$:



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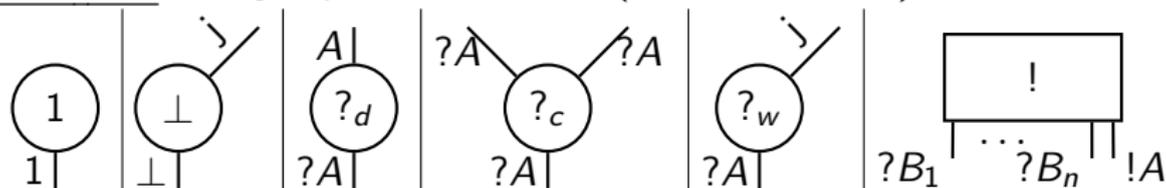
- Proof of the expansion to a unique cut

Same as before: \perp , $?_d$, $?_c$, $?_w$ treated as a \wp , 1 and $!$ irrelevant (like ax)

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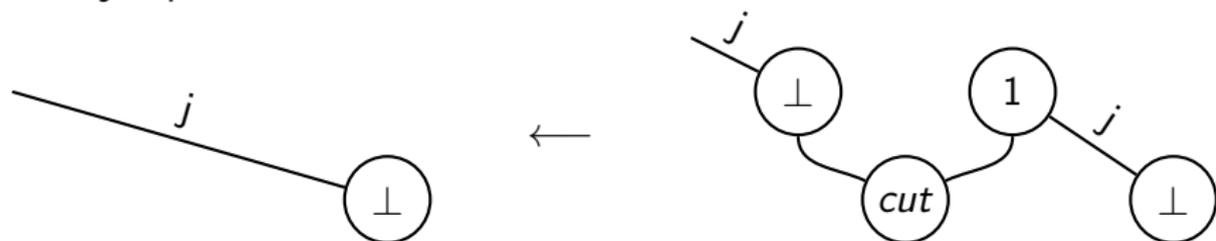


- Proof of the expansion to a unique cut

Same as before: \perp , $?_d$, $?_c$, $?_w$ treated as a \wp , 1 and $!$ irrelevant (like ax)

- Craig's interpolation just needs additional “pre-processing” steps:

- ▶ for a *jump* between A and B :



- ▶ for a $!$ -box using both A and B : a little bit more complex, uses an *induction* (on the net associated to the promotion), then move this *cut* out of the box (by a $?_d - !$ step followed by a $! - !$ step)

In Sequent Calculus for full LL: what are cut-nets?

No proof-net for full LL \rightarrow go to sequent calculus

Definition: Derivation corresponding to a cut-net

A proof derivation π of $\vdash \Gamma$ is a **cut-derivation** if, denoting (A_i) the cut-formulas of π , there is a bipartition $\{B_1, B_2\}$ of $\Gamma \cup (\bigcup_i A_i)$ such that:

- all ax -rules are applied either only on sub-formulas of B_1 or only on sub-formulas of B_2 ; and similarly for $!$ - and \top -rules;
 \rightsquigarrow no ax , $!$ -box or \top using both sides
- given a \perp -rule associated to B_i , the rule just above is not a *cut*-rule and has for principal formula a sub-formula of B_i or is an $ax/!/ \top$ -rule between sub-formulas of B_i ; and similarly for a $?_w$ -rule;
 \rightsquigarrow no \perp - or $?_w$ -jump going from one side to the other
- given an \exists -rule associated to B_i , its witness has no atoms introduced by a \forall -rule associated to B_{1-i} .
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 \rightsquigarrow no \forall -jump going from one side to the other

Deduce Kraig's interpolation as before: introduce a cut when one of the hypothesis of a *cut-derivation* is not respected.

In Sequent Calculus for full LL: proof

Proposition: Expanding to a unique *cut*

Take a *cut*-derivation π . There is a sequence of *cut*-expansion steps yielding a proof with only one *cut*-rule, at its root.

Similar proof as previously: *case study on a last rule*

!! & case: $\vdash A \& B, \Gamma_1, \Gamma_2$ where $\Gamma_2 = \emptyset$ and only one branch has a *cut*:

$$\frac{\frac{\vdash A, \Gamma_1, C_1 \quad \vdash C_1^\perp}{\vdash A, \Gamma_1} \text{ (cut)}}{\vdash A \& B, \Gamma_1} \text{ (&)}$$

How to move the *cut* at the root?

In Sequent Calculus for full LL: proof

Proposition: Expanding to a unique *cut*

Take a *cut-derivation* π . There is a sequence of *cut-expansion steps* yielding a proof with only one *cut-rule*, at its root.

Similar proof as previously: *case study on a last rule*

!!! **& case:** $\vdash A \& B, \Gamma_1, \Gamma_2$ where $\Gamma_2 = \emptyset$ and only one branch has a *cut*:

$$\frac{\frac{\vdash A, \Gamma_1, C_1 \quad \vdash C_1^\perp}{\vdash A, \Gamma_1} \text{ (cut)}}{\vdash A \& B, \Gamma_1} \text{ (&)}$$

How to move the *cut* at the root?

- Introduce a *cut* on $\perp - 1$ in the right branch: *no more stable by fragment*; or
 - Eliminate the *cut* in the left branch: *no more only cut-expansion*.
- ~> The additive connectives need the multiplicative units to work well, but the multiplicative connectives do not! (Quantifiers also need them.)

Conclusion

- Simple visual proof of expansion to a unique *cut*, with and without mix
- Yields a proof of Craig's interpolation for unit-free MLL, MLL and MELL
- Useful for denotational semantic when typing an interface
- Adaptable in sequent calculus for full linear logic, but definitions less visual and some problems with $\&$ and quantifiers

Conclusion

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Thank you!

References I

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- [FOS25] Guido Fiorillo, Daniel Osorio Valencia, and Alexis Saurin. “On Correctness, Sequentialization and Interpolation”. In: *Ninth International Workshop on Trends in Linear Logic and Applications 2025*. Ed. by Lionel Vaux Auclair. 2025. URL: <https://lipn.univ-paris13.fr/TLLA/2025/abstracts/14-fiorillo-osoriovalencia-saurin.pdf>.

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- [Sau25] Alexis Saurin. “Interpolation as Cut-Introduction: On the Computational Content of Craig-Lyndon Interpolation”. In: *International Conference on Formal Structures for Computation and Deduction (FSCD)*. Ed. by Maribel Fernández. Vol. 337. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, July 2025, 32:1–32:21. DOI: 10.4230/LIPIcs.FSCD.2025.32. URL: <https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.FSCD.2025.32>.