Identity of formulas and proofs

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Plan

- **Equality of proofs / terms**
  - In λ-calculus
  - In Linear Logic
- **Equality of formulas / types**
  - In λ-calculus
  - In Linear Logic

Simply typed λ-calculus

Terms

$$M, N \;:=\; x \mid \lambda x.M \mid M\ N$$

Types

$$A, B \;:=\; O \mid A \to B$$

β-reduction

$$(\lambda x.M)\ N \xrightarrow{\beta} M[N/x]$$

η-expansion

$$M \xrightarrow{\eta} (\lambda x.M\ x)$$

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Usually, *syntactic equality* is not enough:

- Quotient in category/denotational model: $M =_{\beta\eta} N \implies [\![M]\!] = [\![N]\!]$
- Church encoding: $\underline{n} := \lambda f.\lambda x.\overbrace{f \ f \ \ldots \ f}^{n \text{ times}} \ x$
  $\underline{2} + \underline{2}$ should be equivalent to $\underline{2} + (\underline{1} + \underline{1})$

$\longrightarrow$ the "good" notion of equality of terms is up to computations (*i.e.* β and η equivalence closures, $=_{\beta\eta}$)

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$\longrightarrow$ the "good" notion of equality of terms is up to computations (*i.e.* β and η equivalence closures, $=_{\beta\eta}$)

Here: focus ***only on equality up to β-reduction*** to simplify

Checking equality of terms

Problem:

- $M =_\beta N$? Give a sequence of terms $M \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} N$
- $M \neq_\beta N$? Prove such a sequence cannot exist!

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Key results:

- β is **strongly normalizing**
  (no infinite sequence of reductions)
- β is **confluent**

$$\begin{tikzcd} & M \arrow[dl, "*"'] \arrow[dr, "*"] & \\ N_1 \arrow[dr, red, "*"'] & & N_2 \arrow[dl, red, "*"] \\ & O & \end{tikzcd}$$

Corollary

$$M =_\beta N \iff \beta(M) = \beta(N) \qquad \textit{where } = \textit{ is syntactic equality!}$$

with $\beta(\cdot)$ the unique normal form of the term

Examples

$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \xleftarrow{\beta^*} \underline{2} + (\underline{1} + \underline{1}) \qquad \underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \neq \underline{3} \xleftarrow{\beta^*} \underline{2} + \underline{1}$

Linear Logic

Formulas

$$
\begin{array}{lll}
A, B \ \coloneqq & \mid X \mid X^{\perp} & \text{(atom)} \\
& \mid A \parr B \mid A \otimes B \mid \perp \mid 1 & \text{(multiplicative)} \\
& \mid A \,\&\, B \mid A \oplus B \mid \top \mid 0 & \text{(additive)} \\
& \mid ?A \mid !A & \text{(exponential)} \\
& \mid \forall X A \mid \exists X A & \text{(quantifier)}
\end{array}
$$

Orthogonality

$$
(X^{\perp})^{\perp} = X \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad \dots
$$

Rules (16)

$$
\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \dots
$$

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Computations: through the Curry-Howard isomorphism, β-reduction corresponds to cut-elimination and η-expansion to axiom-expansion

Cut-elimination

Key steps (9) *"true" computations*

$$\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \overset{\pi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(cut) \quad \xrightarrow{\beta} \quad \overset{\pi}{\vdash A, \Gamma}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, A^{\perp}, \Gamma}}{\vdash B^{\perp} \parr A^{\perp}, \Gamma}\,(\parr) \qquad \dfrac{\overset{\rho}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B^{\perp}, A^{\perp}, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^{\perp}, \Gamma, \Sigma}\,(cut) \qquad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

Commutative steps (15) *used to reach a key step*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, C, \Gamma}}{\vdash A^{\perp}, B \parr C, \Gamma}\,(\parr) \qquad \overset{\rho}{\vdash A, \Delta}}{\vdash B \parr C, \Gamma, \Delta}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A^{\perp}, B, C, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B, C, \Gamma, \Delta}\,(cut)}{\vdash B \parr C, \Gamma, \Delta}\,(\parr)$$

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Cut-elimination on an example

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash A^\perp, A \otimes B, B^\perp}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(cut)$$

$$\xrightarrow{\beta_{com}}$$

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Checking equality of proofs

<u>Problem:</u>

- $\pi =_\beta \rho$? Give a sequence of proofs $\pi \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} \rho$
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Can we do the same as in λ-calculus?

- Cut-elimination is **strongly normalizing**?

- Cut-elimination is **confluent**?

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- Cut-elimination is **confluent**?
  Not at all!

Cut-elimination is not confluent!

$$
\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \quad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A \otimes B, A^\perp, B^\perp}\,({\color{green}\otimes}) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \quad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A^\perp \otimes C^\perp, A, C}\,({\color{orange}\otimes})}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(cut)
$$

$$
\swarrow \beta^* \qquad\qquad \searrow \beta^*
$$

$$
\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \quad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A, A^\perp \otimes C^\perp, C}\,({\color{orange}\otimes}) \quad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,({\color{green}\otimes})
\quad \neq \quad
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$$

Irreversible choice at the beginning: first commutative case with the left $\otimes$-rule or with the right one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, . . .

Cut-elimination is not confluent!

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$$\swarrow_{\beta^*} \qquad\qquad\qquad {}_{\beta^*}\searrow$$

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No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, ...

But confluence up to rule commutation!

Idea

$(a + b) \times (c + d)$ reduces by distributivity laws to both $(a \times c + a \times d) + (b \times c + b \times d)$ and $(a \times c + b \times c) + (a \times d + b \times d)$ which are equal *up to* associativity and commutativity of $+$.

Rule commutations (from a list of cases)

$$\dfrac{\overset{\pi}{\vdash C,\Gamma} \qquad \dfrac{\overset{\rho}{\vdash A,D,\Delta} \quad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B, D,\Delta,\Sigma}\,(\otimes)}{\vdash A\otimes B, C\otimes D,\Gamma,\Delta,\Sigma}\,(\otimes) \quad\equiv\quad \dfrac{\dfrac{\overset{\pi}{\vdash C,\Gamma}\quad \overset{\rho}{\vdash A,D,\Delta}}{\vdash A, C\otimes D,\Gamma,\Delta}\,(\otimes) \qquad \overset{\tau}{\vdash B,\Sigma}}{\vdash A\otimes B, C\otimes D,\Gamma,\Delta,\Sigma}\,(\otimes)$$

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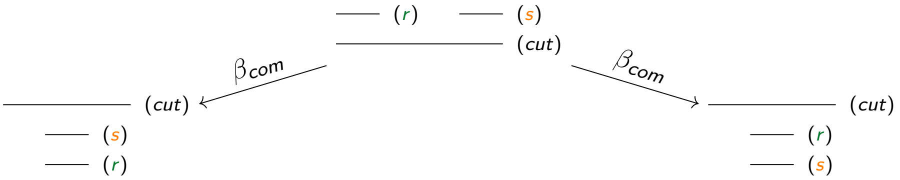
$$\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{\overset{\rho}{\vdash B,C,\Delta} \quad \overset{\tau}{\vdash B,D,\Delta}}{\vdash B, C\,\&\,D,\Delta}\,(\&)}{\vdash A\otimes B, C\,\&\,D,\Gamma,\Delta}\,(\otimes) \quad\equiv\quad \dfrac{\dfrac{\overset{\pi}{\vdash A,\Gamma}\quad \overset{\rho}{\vdash B,C,\Delta}}{\vdash A\otimes B, C,\Gamma,\Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A,\Gamma}\quad \overset{\tau}{\vdash B,D,\Delta}}{\vdash A\otimes B, D,\Gamma,\Delta}\,(\otimes)}{\vdash A\otimes B, C\,\&\,D,\Gamma,\Delta}\,(\&)$$

... (and many many many more)

Rule commutations (from a general method)

Every pair $\begin{array}{l}\overline{}\ (s)\\ \overline{}\ (r)\end{array} \equiv \begin{array}{l}\overline{}\ (r)\\ \overline{}\ (s)\end{array}$ coming from:



Approximately N^2 commutations with N the number of rules $\longrightarrow$ 93 in LL!

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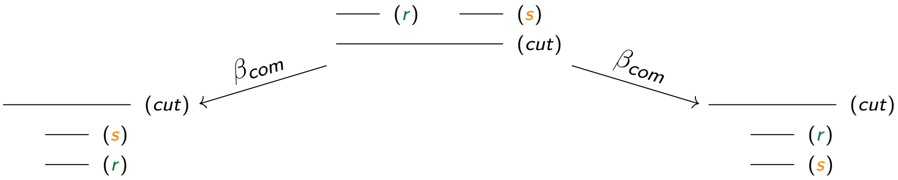

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Remarks

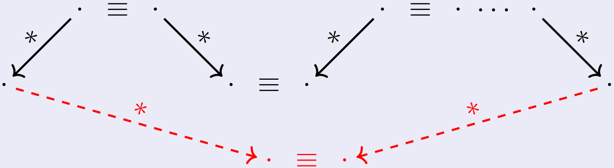
- $\equiv\ \subseteq\ =_\beta$ trivially
- $\equiv$ is exactly the usual rule commutations without the $! - ?_c$ and $! - ?_w$ commutations!

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad\not\equiv\quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \quad\text{and}\quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad\not\equiv\quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

Confluence up to rule commutation

Theorem (Proved in MALL [DL23] – also on a term syntax in [CP05]; in progress for full LL)

*Cut-elimination is **Church-Rosser modulo** rule commutation.*



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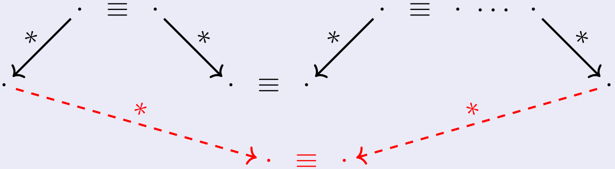

Theorem 2.2 from [AT12]

Let $\vdash\!\dashv$, $\rightarrow$ *and* $\rightsquigarrow$ *be relations such that* $\vdash\!\dashv$ *is symmetric and* $\rightsquigarrow \subseteq \vdash\!\dashv$.
Set $\Rightarrow = \rightarrow \cup \rightsquigarrow$. *Suppose:*

1. $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow \subseteq \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$
3. $\vdash\!\dashv \cdot \rightarrow \subseteq (\overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow) \cup (\rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow)$

Then $\rightarrow$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$.

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\cdot \arrow[drrrr, dashed, "*"] & & & & \cdot & \equiv & \cdot & & & & & & \cdot \arrow[dllllll, dashed, "*"'] \\
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\end{tikzcd}
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\cdot \arrow[drrrr, dashed, red, "*"] & & & & \cdot & \equiv & \cdot & & & & & \cdot \arrow[dlllll, dashed, red, "*"'] \\
 & & & & \color{red}\cdot & \color{red}\equiv & \color{red}\cdot & & & & &
\end{tikzcd}
$$

Corollary: Confluence up to rule commutation

If π_1 and π_2 are cut-free proofs obtained by cut-elimination from a same proof π, then $\pi_1 \equiv \pi_2$.

$$
\begin{tikzcd}
 & \pi \arrow[dl, "*"'] \arrow[dr, "*"] & \\
\pi_1 & \color{red}\equiv & \pi_2
\end{tikzcd}
$$

Corollary: Equality on normal forms

Between cut-free proofs, $=_\beta$ is exactly $\equiv$.

Consequences

- **"Bureaucracy":** have to order all rules in sequent calculus, but some order does not matter and no canonical choice

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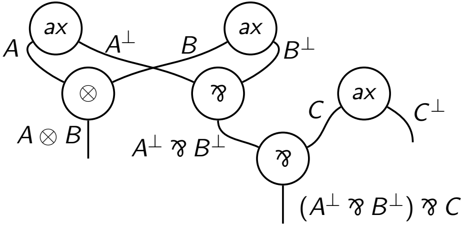
- **Proof-nets:** identify proofs exactly up to rule commutation $\equiv$
  $\longrightarrow$ $=_\beta$ is simply equality of graphs (for normal forms)!
  $\longrightarrow$ cut-elimination is confluent in proof-nets!
  $\longrightarrow$ some difficulties for $! - ?_c$ and $! - ?_w$



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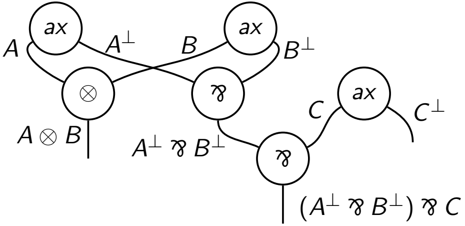

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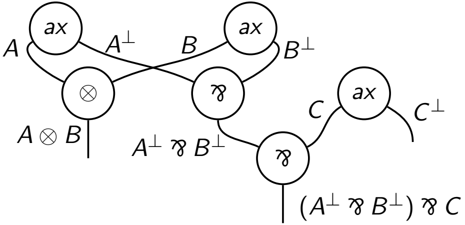

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- **Axiom-expansion** to take into account, along with its interactions with cut-elimination (*e.g.* commutations as rewriting systems) . . .

Interlude: What about Classical Logic?

Cut-elimination equalizes **all proofs** of a same sequent (by Lafont in [GLT89, Appendix B.1]): $=_\beta$ is the largest possible!

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A}}{\vdash C, A}\,(w) \qquad \dfrac{\overset{\rho}{\vdash A}}{C \vdash A}\,(w)}{\vdash A, A}\,(cut)}{\vdash A}\,(c)$$

$$\swarrow_{\beta} \qquad\qquad\qquad\qquad \searrow_{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A}}{\vdash A, A}\,(w)}{\vdash A}\,(c) \qquad\qquad\qquad\qquad \dfrac{\dfrac{\overset{\rho}{\vdash A}}{\vdash A, A}\,(w)}{\vdash A}\,(c)$$

$$\sim \qquad\qquad\qquad\qquad\qquad\qquad \sim$$

$$\overset{\pi}{\vdash A} \qquad\qquad\qquad\qquad\qquad\qquad \overset{\rho}{\vdash A}$$

Plan

- **Equality of proofs / terms**
  - In λ-calculus
  - In Linear Logic
- **Equality of formulas / types**
  - In λ-calculus
  - In Linear Logic

Equality of formulas as Isomorphisms

(Type) Isomorphisms relate types/formulas/objects which are "the same", *i.e.* which are indistinguishable.

Generally in **category theory**:

$$\begin{tikzcd}
A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] \arrow[r, phantom, "A \simeq B"] & B \arrow[l, bend left, "g"] \arrow[loop right, "id_B"]
\end{tikzcd}$$

Instantiation in λ-calculi, logics, ...

Equality of formulas as Isomorphisms

(Type) Isomorphisms relate types/formulas/objects which are "the same", *i.e.* which are indistinguishable.

Generally in **category theory**:

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[rr, bend left, "f"] \arrow[rr, phantom, "A \simeq B"] & & B \arrow[loop right, "id_B"] \arrow[ll, bend left, "g"] \end{tikzcd}$$

Instantiation in λ-calculi, logics, ...

General question: give a **caracterisation** of all isomorphisms in a category

$\longrightarrow$ an equational theory = basic isomorphisms from which all others follow

Soundness $\longrightarrow$ Easy (but tedious) work.

Completeness $\longrightarrow$ Two main approaches:

Syntactic the analysis of pairs of terms composing to the identity should provide information on their type (if $=$ is simple)

Semantic find a model with the same isomorphisms than in the syntax but where they can be computed more easily (typically reducing to equality between combinatorial objects)

Isomorphisms in λ-calculus

Isomorphism $A \simeq B$

Terms M of $A \to B$ and N of $B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x \quad \text{and} \quad M \circ N =_{\beta\eta} \lambda x^B.x$$

Examples

- $A \to (B \to C) \simeq B \to (A \to C)$
  with $M = \lambda f.\lambda b.\lambda a.(f\ a)\ b$
  and $N = \lambda f.\lambda a.\lambda b.(f\ b)\ a$
- $A \times B \simeq B \times A$
  with $M = N = \lambda c.(\pi_2\ c, \pi_1\ c)$
- $(A \times B) \to C \simeq A \to (B \to C)$

Isomorphisms in λ-calculus

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- $A \to (B \to C) \simeq B \to (A \to C)$
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  and $N = \lambda f.\lambda a.\lambda b.(f\ b)\ a$
- $A \times B \simeq B \times A$
  with $M = N = \lambda c.(\pi_2\ c, \pi_1\ c)$
- $(A \times B) \to C \simeq A \to (B \to C)$

Definition **simple** but hard to use practically:

→ Tedious to prove $A \simeq B$: give terms M and N then show both compositions $=_{\beta\eta}$ the identity.

→ Horrible to prove $A \not\simeq B$: argue for every terms M and N, one composition $\neq_{\beta\eta}$ the identity.

Isomorphisms in λ-calculus

Isomorphism $A \simeq B$

Terms M of $A \to B$ and N of $B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x \quad \text{and} \quad M \circ N =_{\beta\eta} \lambda x^B.x$$

For λ-calculus with products and unit type / cartesian closed categories **Semantic (finite sets) [Sol83]**

$\times$	$A \times (B \times C) \simeq (A \times B) \times C$	$A \times B \simeq B \times A$
$\times$ and $\to$	$(A \times B) \to C \simeq A \to (B \to C)$	$A \to (B \times C) \simeq (A \to B) \times (A \to C)$
1	$1 \times A \simeq A$ $\quad 1 \to A \simeq A$	$A \to 1 \simeq 1$

Reduction to Tarski's High School Algebra Problem: can all equalities involving product, exponential and 1 be found using only

$$a(bc) = (ab)c \qquad ab = ba$$

$$c^{ab} = (c^b)^a \qquad (bc)^a = b^a c^a$$

$$1a = a \qquad a^1 = a \qquad 1^a = 1$$

Isomorphisms in λ-calculus

Isomorphism $A \simeq B$

Terms M of $A \to B$ and N of $B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x \quad \text{and} \quad M \circ N =_{\beta\eta} \lambda x^B.x$$

For λ-calculus with products, unit type and sums / cartesian closed categories with binary coproducts [FDB02]

NOT FINITELY AXIOMATISABLE

Give an <u>infinite</u> family of isomorphisms that cannot be obtained by any <u>finite</u> number of isomorphisms.

Interlude: What about Classical Logic?

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\ (cut) =_{\beta\eta} \dfrac{}{A \vdash A}\ (ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\ (cut) =_{\beta\eta} \dfrac{}{B \vdash B}\ (ax)$$

Reminder

Cut-elimination equalizes all proofs of a same sequent.

Fact

$$A \simeq B \iff A \dashv\vdash B$$

Not very exiting, but **not trivial**: deciding $A \dashv\vdash B$ is equivalent to deciding provability!

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall XA \,\&\, \forall XB$ $\forall XA \parr B \simeq \forall X(A \parr B)^*$	$\exists X(A \oplus B) \simeq \exists XA \oplus \exists XB$ $\exists XA \otimes B \simeq \exists X(A \otimes B)^*$	$\forall X \top \simeq \top$ $\forall X \forall Y A \simeq \forall Y \forall X A$	$\exists X 0 \simeq 0$ $\exists X \exists Y A \simeq \exists Y \exists X A$

$*$ **if X not free in B**

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\ (cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\ (ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\ (cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\ (ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\ !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\ ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall XA \,\&\, \forall XB$ $\forall XA \parr B \simeq \forall X(A \parr B)^*$	$\exists X(A \oplus B) \simeq \exists XA \oplus \exists XB$ $\exists XA \otimes B \simeq \exists X(A \otimes B)^*$	$\forall X \top \simeq \top$ $\forall X \forall YA \simeq \forall Y \forall XA$	$\exists X0 \simeq 0$ $\exists X \exists YA \simeq \exists Y \exists XA$

$*$ **if X not free in B**

Semantic method complicated: most models come with $\top \otimes A \simeq \top \otimes B$ while in the syntax $\top \otimes A \not\simeq \top \otimes B$

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

For Multiplicative Linear Logic / $\star$-autonomous categories

Syntactic (proof-nets) [BD99]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall XA \,\&\, \forall XB$ $\forall XA \parr B \simeq \forall X(A \parr B)^*$	$\exists X(A \oplus B) \simeq \exists XA \oplus \exists XB$ $\exists XA \otimes B \simeq \exists X(A \otimes B)^*$	$\forall X\top \simeq \top$ $\forall X\forall YA \simeq \forall Y\forall XA$	$\exists X0 \simeq 0$ $\exists X\exists YA \simeq \exists Y\exists XA$

$*$ if X not free in B

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

For Multiplicative-Additive Linear Logic / $\star$-autonomous categories with finite products **Syntactic (proof-nets) [DL23]**

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall XA \,\&\, \forall XB$ $\forall XA \parr B \simeq \forall X(A \parr B)^*$	$\exists X(A \oplus B) \simeq \exists XA \oplus \exists XB$ $\exists XA \otimes B \simeq \exists X(A \otimes B)^*$	$\forall X\top \simeq \top$ $\forall X\forall YA \simeq \forall Y\forall XA$	$\exists X0 \simeq 0$ $\exists X\exists YA \simeq \exists Y\exists XA$

$*$ if X not free in B

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

For Polarized Linear Logic

Semantic (games, forest isomorphisms) [Lau05]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall XA \,\&\, \forall XB$ $\forall XA \parr B \simeq \forall X(A \parr B)^*$	$\exists X(A \oplus B) \simeq \exists XA \oplus \exists XB$ $\exists XA \otimes B \simeq \exists X(A \otimes B)^*$	$\forall X \top \simeq \top$ $\forall X \forall Y A \simeq \forall Y \forall X A$	$\exists X 0 \simeq 0$ $\exists X \exists Y A \simeq \exists Y \exists X A$

$*$ if X not free in B

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\ (cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\ (ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\ (cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\ (ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
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Quantifiers	$\forall X(A \& B) \simeq \forall XA \& \forall XB$ $\forall XA \parr B \simeq \forall X(A \parr B)^*$	$\exists X(A \oplus B) \simeq \exists XA \oplus \exists XB$ $\exists XA \otimes B \simeq \exists X(A \otimes B)^*$	$\forall X\top \simeq \top$ $\forall X \forall YA \simeq \forall Y \forall XA$	$\exists X0 \simeq 0$ $\exists X \exists YA \simeq \exists Y \exists XA$

$*$ if X not free in B

Natural perspectives:

- MELL
- M(A)LL with quantifiers

Retractions in Linear Logic

Retraction $A \trianglelefteq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Example

$$A \trianglelefteq (A \multimap A) \otimes A$$

No conjecture, even in MLL!

Indecidable in propositional LL!

Thank you!

References I

[AT12] Takahito Aoto and Yoshihito Toyama. "A Reduction-Preserving Completion for Proving Confluence of Non-Terminating Term Rewriting Systems". In: *Logical Methods in Computer Science* 8.1 (Mar. 2012), pp. 1–29. DOI: `10.2168/LMCS-8(1:31)2012`.

[BD99] Vincent Balat and Roberto Di Cosmo. "A Linear Logical View of Linear Type Isomorphisms". In: *Computer Science Logic*. Ed. by Jörg Flum and Mario Rodríguez-Artalejo. Vol. 1683. Lecture Notes in Computer Science. Springer, 1999, pp. 250–265.

References II

[CP05] Robin Cockett and Craig Pastro. "A Language For Multiplicative-additive Linear Logic". In: *Electronic Notes in Theoretical Computer Science* 122 (2005). Proceedings of the 10th Conference on Category Theory in Computer Science (CTCS 2004), pp. 23–65. DOI: /10.1016/j.entcs.2004.06.049. URL: https://www.sciencedirect.com/science/article/pii/S1571066105000320.

References III

[DL23] Rémi Di Guardia and Olivier Laurent. "Type Isomorphisms for Multiplicative-Additive Linear Logic". In: *International Conference on Formal Structures for Computation and Deduction (FSCD)*. Ed. by Marco Gaboardi and Femke van Raamsdonk. Vol. 260. Leibniz International Proceedings in Informatics (LIPIcs). Full version with proofs on `https://hal.science/hal-04082204`. Schloss Dagstuhl – Leibniz-Zentrum fuer Informatik, July 2023, 26:1–26:21. DOI: `10.4230/LIPIcs.FSCD.2023.26`.

[FDB02] Marcelo Fiore, Roberto Di Cosmo, and Vincent Balat. "Remarks on Isomorphisms in Typed Lambda Calculi with Empty and Sum Types". In: *Proceedings of the seventeenth annual symposium on Logic In Computer Science*. IEEE. Copenhagen: IEEE Computer Society Press, July 2002, pp. 147–156.

References IV

[GLT89] Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types*. Cambridge tracts in theoretical computer science 7. Cambridge University Press, 1989.

[HH16] Willem Heijltjes and Robin Houston. "Proof equivalence in MLL is PSPACE-complete". In: *Logical Methods in Computer Science* 12.1 (2016). DOI: `10.2168/LMCS-12(1:2)2016`.

[Lau05] Olivier Laurent. "Classical isomorphisms of types". In: *Mathematical Structures in Computer Science* 15.5 (Oct. 2005), pp. 969–1004.

[Lin95] Patrick Lincoln. "Deciding provability of linear logic formulas". In: *Advances in Linear Logic*. Ed. by Jean-Yves Girard, Yves Lafont, and Laurent Regnier. Vol. 222. London Mathematical Society Lecture Note Series. Cambridge University Press, 1995, pp. 109–122.

References V

[Sol83] Sergei Soloviev. "The category of finite sets and cartesian closed categories". In: *Journal of Soviet Mathematics* 22.3 (1983), pp. 1387–1400.

Retractions and Provability

Fact

$$!X \trianglelefteq\, !X \otimes !(X \otimes A) \iff \vdash A \text{ is provable}$$

$$X \trianglelefteq X \,\&\, (X \otimes A) \iff \vdash A \text{ is provable}$$

$$A \trianglelefteq A \oplus B \iff \vdash B^{\perp}, A \text{ is provable}$$

Fragment	Provability
LL	Undecidable ☹
MELL	TOWER-hard ☹ (decidability is open)
MALL	PSPACE-complete ☹
ALL	P-complete

(an overview of these results on provability can be found in [Lin95])