Identity of Proofs and Formulas in Linear Logic

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Plan

- **Equality of proofs / terms**
  - In λ-calculus
  - In Linear Logic
- **Equality of formulas / types**
  - In λ-calculus
  - In Linear Logic
    - Isomorphisms
    - Retractions

Simply typed λ-calculus

Terms

$$M, N \;:=\; x \mid \lambda x.M \mid M\ N$$

Types

$$A, B \;:=\; O \mid A \to B$$

β-**reduction**

$$(\lambda x.M)\ N \xrightarrow{\beta} M[N/x]$$

η-**expansion**

$$M \xrightarrow{\eta} (\lambda x.M\ x)$$

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Syntactic equality is usually not enough:

- **Quotient in category/denotational model:** $M =_{\beta\eta} N \implies [\![M]\!] = [\![N]\!]$
- **Church encoding:** $\underline{n} := \lambda f.\lambda x.\overbrace{f\ f\ \dots\ f}^{n \text{ times}}\ x$
  $\underline{2} + \underline{2}$ should be equivalent to $\underline{2} + (\underline{1} + \underline{1})$

$\longrightarrow$ the "right" notion of equality is up to computations $= \beta\eta$ equivalence

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Here: ***only on equality up to β-reduction*** to simplify

Checking equality of terms

Problem:

- $M =_\beta N$? Give a sequence of terms $M \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} N$
- $M \neq_\beta N$? Prove such a sequence cannot exist!

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Key results:

- β is **strongly normalizing** (no infinite sequence of reductions)
- β is **confluent**

$$\begin{tikzcd} & M \arrow[dl, "*"'] \arrow[dr, "*"] & \\ N_1 \arrow[dr, "*"'] & & N_2 \arrow[dl, "*"] \\ & O & \end{tikzcd}$$

Corollary

$$M =_\beta N \iff \beta(M) = \beta(N)$$

with $\beta(\cdot)$ the unique normal form of the term

Examples

$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \xleftarrow{\beta^*} \underline{2} + (\underline{1} + \underline{1})$ $\quad$ $\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \neq \underline{3} \xleftarrow{\beta^*} \underline{2} + \underline{1}$

Linear Logic

Formulas

$$
\begin{array}{lll}
A, B \;:= & \mid X \mid X^{\perp} & \textit{atom} \\
& \mid A \parr B \mid A \otimes B \mid \perp \mid 1 & \textit{multiplicative} \\
& \mid A \,\&\, B \mid A \oplus B \mid \top \mid 0 & \textit{additive} \\
& \mid ?A \mid !A & \textit{exponential} \\
& \mid \forall X\ A \mid \exists X\ A & \textit{quantifier}
\end{array}
$$

Involutive Negation / Orthogonality

$$(X^{\perp})^{\perp} = X$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} \quad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \quad \perp^{\perp} = 1 \quad 1^{\perp} = \perp$$

$$\dots$$

Sub-systems

- MLL = atom + multiplicative
- MALL = atom + multiplicative + additive
- ...

16 Rules of Linear Logic

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \mathbin{\&} B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

$$X \text{ not free in } \Gamma \ \ \frac{\vdash A, \Gamma}{\vdash \forall X\ A, \Gamma}\ (\forall) \qquad \frac{\vdash A[B/X], \Gamma}{\vdash \exists X\ A, \Gamma}\ (\exists)$$

16 or up to 20 Rules of Linear Logic

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$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ (mix_2) \qquad \frac{}{\vdash}\ (mix_0) \qquad \frac{\vdash \Gamma \quad \vdash \Gamma}{\vdash \Gamma}\ (\cup) \qquad \frac{}{\vdash \Gamma}\ (\emptyset)$$

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Curry-Howard isomorphism: β-reduction $\approx$ cut-elimination
η-expansion $\approx$ axiom-expansion

Cut-elimination

Key steps (9) *"true" computations*

$$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \overset{\pi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(cut) \quad \xrightarrow{\beta} \quad \overset{\pi}{\vdash A, \Gamma}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B^\perp, A^\perp, \Gamma}}{\vdash B^\perp \parr A^\perp, \Gamma}\,(\parr) \qquad \dfrac{\overset{\rho}{\vdash A, \Delta} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B^\perp, A^\perp, \Gamma} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma}\,(cut) \qquad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

Commutative steps (15) *used to reach a key step*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, C, \Gamma}}{\vdash A^\perp, B \parr C, \Gamma}\,(\parr) \qquad \overset{\rho}{\vdash A, \Delta}}{\vdash B \parr C, \Gamma, \Delta}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, C, \Gamma} \qquad \overset{\rho}{\vdash A, \Delta}}{\vdash B, C, \Gamma, \Delta}\,(cut)}{\vdash B \parr C, \Gamma, \Delta}\,(\parr)$$

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Cut-elimination on an example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash B, B^\perp$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash A^\perp, A \otimes B, B^\perp$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A, A^\perp$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash A, A^\perp \otimes C^\perp, C$}
\RightLabel{(cut)}
\BinaryInfC{$\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C$}
\end{prooftree}
$$

$$\xrightarrow{\beta_{com}}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash A^\perp, A$}
\AxiomC{}
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- Cut-elimination is **strongly normalizing**?

- Cut-elimination is **confluent**?

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- Cut-elimination is **confluent**?
  Not at all!

Cut-elimination is not confluent!

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A \otimes B, A^\perp, B^\perp}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A^\perp \otimes C^\perp, A, C}\,(\otimes)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(cut)$$

$$\swarrow \beta^* \qquad\qquad \beta^* \searrow$$

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Irreversible choice at the beginning:

first commutative case with the left $\otimes$-rule or with the right one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, . . .

Cut-elimination is not confluent!

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But confluence up to rule commutation!

Idea

$(a+b)\times(c+d)$ reduces by distributivity laws to both
$(a\times c + a\times d) + (b\times c + b\times d)$ and $(a\times c + b\times c) + (a\times d + b\times d)$
which are equal up to associativity and commutativity of $+$

Rule commutations (from a list of cases)

$$\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \dfrac{\overset{\rho}{\vdash A, D, \Delta} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes) \quad \equiv \quad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \overset{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}\,(\otimes) \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$$

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$$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\rho}{\vdash B, C, \Delta} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash B, C \,\&\, D, \Delta}\,(\&)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes) \quad \equiv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\rho}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&)$$

... and many many many more ...

Rule commutations (from a general method)

Every pair $\dfrac{\dfrac{}{\quad}\,(s)}{\quad}\,(r) \equiv \dfrac{\dfrac{}{\quad}\,(r)}{\quad}\,(s)$ coming from:

$$\begin{tikzcd}
 & \dfrac{\dfrac{}{\quad}\,(r) \qquad \dfrac{}{\quad}\,(s)}{\qquad}\,(cut) \arrow[dl, "\beta_{com}"'] \arrow[dr, "\beta_{com}"] & \\
\dfrac{\dfrac{\dfrac{}{\quad}\,(s)}{\quad}\,(r)}{\qquad}\,(cut) & & \dfrac{\dfrac{\dfrac{}{\quad}\,(r)}{\quad}\,(s)}{\qquad}\,(cut)
\end{tikzcd}$$

Approximately N^2 commutations with N the number of rules $\longrightarrow$ 93 in LL!

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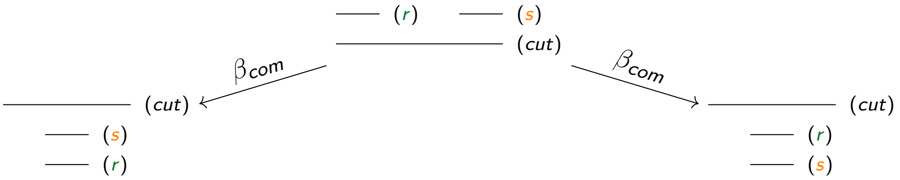


Approximately N^2 commutations with N the number of rules $\longrightarrow$ 93 in LL!

Remarks

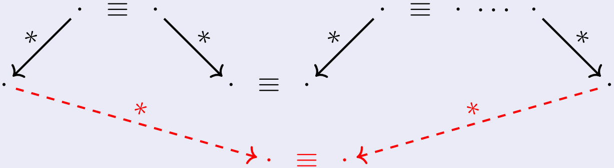
- $\equiv\, \subseteq\, =_\beta$ trivially
- $\equiv$ is exactly the usual (cut-free) rule commutations without the $! - ?_c$ and $! - ?_w$ commutations!

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \;\not\equiv\; \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \quad \text{and} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \;\not\equiv\; \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

Confluence up to rule commutation

Theorem (Proved in MALL [DL23] – also on a term syntax in [CP05]; in progress for full LL)

*Cut-elimination is **Church-Rosser modulo** rule commutation.*



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$$\begin{tikzcd}
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\cdot \arrow[drrrrr, dashed, red, "*"] & & & & \cdot & \equiv & \cdot & & & & & & \cdot \arrow[dlllll, dashed, red, "*"'] \\
 & & & & & \color{red}\cdot & \color{red}\equiv & \color{red}\cdot & & & & &
\end{tikzcd}$$

Theorem 2.2 from [AT12]

Let $\vdash\!\dashv$, $\rightarrow$ *and* $\rightsquigarrow$ *be relations such that* $\vdash\!\dashv$ *is symmetric and* $\rightsquigarrow \subseteq \vdash\!\dashv$.
Set $\Rightarrow = \rightarrow \cup \rightsquigarrow$. *Suspose:*

1. $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow \subseteq \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$
3. $\vdash\!\dashv \cdot \rightarrow \subseteq \ (\overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow) \cup (\rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow)$

Then $\rightarrow$ *is Church-Rosser modulo* $\overset{*}{\vdash\!\dashv}$.

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\cdot \arrow[drrrr, dashed, red, "*"] & & & & \cdot & \equiv & \cdot & & & & \cdot \arrow[dllll, dashed, red, "*"'] \\
 & & & & \color{red}\cdot & \color{red}\equiv & \color{red}\cdot & & & &
\end{tikzcd}$$

Theorem 2.2 from [AT12]

Let $\vdash\!\dashv$, $\rightarrow$ and $\rightsquigarrow$ be relations such that $\vdash\!\dashv$ is symmetric and $\rightsquigarrow \subseteq \vdash\!\dashv$. Set $\Rightarrow = \rightarrow \cup \rightsquigarrow$. Suppose:

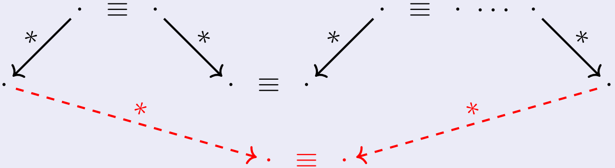
1. $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow \subseteq \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$
3. $\vdash\!\dashv \cdot \rightarrow \subseteq (\overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow) \cup (\rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow)$

Then $\rightarrow$ is Church-Rosser modulo $\overset{*}{\vdash\!\dashv}$.

Confluence up to rule commutation

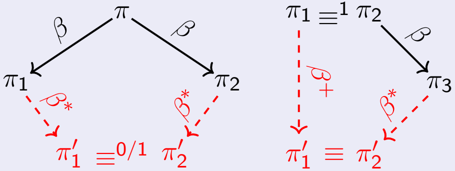
Theorem (Proved in MALL [DL23] – also on a term syntax in [CP05]; in progress for full LL)

Cut-elimination is ***Church-Rosser modulo*** *rule commutation.*



Proof.

#(cut steps)^2
$\sim N^2$ cases



#(cut steps)$\times$
#(communtations)
$\sim N^3$ cases

(with N the number of rules in the logic) □

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Theorem (Proved in MALL [DL23] – also on a term syntax in [CP05]; in progress for full LL)

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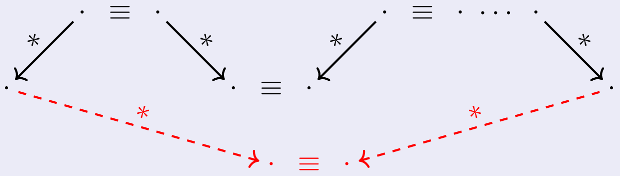

A profusion of thousands of similar cases to check
$\longrightarrow$ horrible and imprecise on paper, to formalize in a **proof assistant**!

But no adequate existing library!

The **exchange** rule overcomplicates, want to be up to exchange while able to distinguish **occcurrences**; which $\bot$ is kept in $\dfrac{\vdash \Gamma, \bot}{\vdash \Gamma, \bot, \bot}\ (\bot)$?

Confluence up to rule commutation

Theorem (Proved in MALL [DL23] – also on a term syntax in [CP05]; in progress for full LL)

*Cut-elimination is **Church-Rosser modulo** rule commutation.*

$$\begin{tikzcd}
 & \cdot \arrow[dl, "*"'] & \equiv & \cdot \arrow[dr, "*"] & & & & \cdot \arrow[dl, "*"'] & \equiv & \cdot\ \cdots & \cdot \arrow[dr, "*"] & \\
\cdot \arrow[drrrr, dashed, red, "*"] & & & & \cdot & \equiv & \cdot & & & & & \cdot \arrow[dlllll, dashed, red, "*"'] \\
 & & & & \cdot & \color{red}{\equiv} & \cdot & & & & &
\end{tikzcd}$$

Corollary: Confluence up to rule commutation

If π_1 and π_2 are cut-free proofs obtained by cut-elimination from a same proof π, then $\pi_1 \equiv \pi_2$.

$$\begin{tikzcd}
 & \pi \arrow[dl, "*"'] \arrow[dr, "*"] & \\
\pi_1 & \color{red}{\equiv} & \pi_2
\end{tikzcd}$$

Corollary: Equality on normal forms

Between cut-free proofs, $=_\beta$ is exactly $\equiv$.

Consequences & Avail

- **“Bureaucracy”:** have to order all rules in sequent calculus, but some order does not matter and no canonical choice

Consequences & Avail

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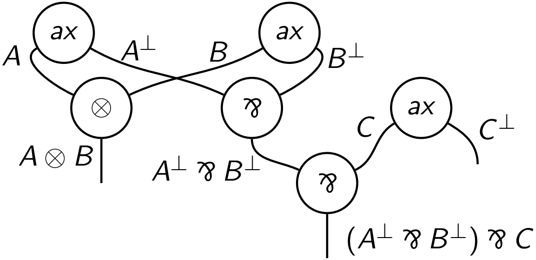
- **Proof-nets:** identify proofs exactly up to rule commutation $\equiv$
  - $=_\beta$ is equality of graphs (on normal forms)
  - cut-elimination is confluent
  - difficulties for $! - ?_c$ and $! - ?_w$
  - defined only in some sub-systems of LL



Consequences & Avail

- **“Bureaucracy”:** have to order all rules in sequent calculus, but some order does not matter and no canonical choice
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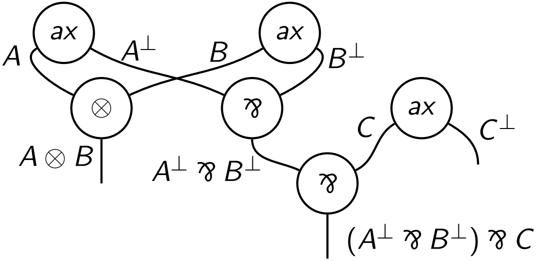

- **Complexity:** rule commutation is easier than $=_\beta$ but is NOT “nice”
  - deciding equivalence of MLL proofs is PSPACE-complete [HH16]
  - equivalence of LL proofs is undecidable:

$$\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \equiv \quad \dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \text{ is provable}$$

But we have more than cut-elimination . . .

- **Axiom-expansion** to take into account, along with its *interactions* with cut-elimination (*e.g.* commutations as rewriting systems)

$$\begin{tikzcd} \pi \arrow[r, "\eta^*"] \arrow[d, "\beta^*"] & \tau \arrow[d, dashed, "\beta^*"] & \eta\text{-normal} \\ \phi \arrow[r, dashed, "\eta^*"] & \mu & \eta\text{-normal} \end{tikzcd}$$

(which holds without 2nd order quantifiers)

But we have more than cut-elimination . . .

- **Axiom-expansion** to take into account, along with its *interactions* with cut-elimination (*e.g.* commutations as rewriting systems)
- One may want **more commutations**, yielding even more cases!

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad \equiv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad \equiv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \quad \equiv \quad \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \quad \text{when } \pi \text{ is “witness irrelevant”}$$

But we have more than cut-elimination . . .

- **Axiom-expansion** to take into account, along with its *interactions* with cut-elimination (*e.g.* commutations as rewriting systems)
- One may want **more commutations**, yielding even more cases!
- One may also want **other rewritings**, for which to check *interactions* with cut-elimination and axiom-expansion

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(?_c) \quad \rightsquigarrow \quad \overset{\pi}{\vdash ?A, \Gamma}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash}\,(mix_0)}{\vdash \Gamma}\,(mix_2) \quad \rightsquigarrow \quad \overset{\pi}{\vdash \Gamma}$$

$$\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash \Gamma}\,(\emptyset)}{\vdash \Gamma}\,(\cup) \quad \rightsquigarrow \quad \overset{\pi}{\vdash \Gamma}$$

Plan

- Equality of proofs / terms
  - In λ-calculus
  - In Linear Logic
- Equality of formulas / types
  - In λ-calculus
  - In Linear Logic
    - Isomorphisms
    - Retractions

Equality of formulas as Isomorphisms

(Type) Isomorphisms relate indistinguishable types/formulas/objects
Generally in **category theory**:

$$\begin{tikzcd}
A \arrow[loop left, "id_A"] \arrow[rr, bend left, "f"] \arrow[rr, phantom, "A \simeq B"] & & B \arrow[ll, bend left, "g"] \arrow[loop right, "id_B"]
\end{tikzcd}$$

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Definition **simple** but hard to use in practice

Problem: **caracterise** exactly the isomorphisms of a category
$\hookrightarrow$ give an **equational theory** $=$ basic isomorphisms generating all others
Soundness $\longrightarrow$ Easy but tedious
Completeness $\longrightarrow$ Two main approaches:

Syntactic analyse pairs of terms composing to id to deduce their types
$\rightsquigarrow$ need $=$ simple

Semantic do it in a model with no more isomorphisms and a simpler $=$
$\rightsquigarrow$ need a far yet close model

Isomorphisms in λ-calculus

Isomorphism $A \simeq B$

Terms M of $A \to B$ and N of $B \to A$ such that

$$N \circ M =_{\beta\eta} \lambda x^A.x \quad \text{and} \quad M \circ N =_{\beta\eta} \lambda x^B.x$$

Examples

- $A \to (B \to C) \simeq B \to (A \to C)$
  with $M = \lambda f.\lambda b.\lambda a.(f\ a)\ b$
  and $N = \lambda f.\lambda a.\lambda b.(f\ b)\ a$
- $A \times B \simeq B \times A$
  with $M = N = \lambda c.(\pi_2\ c, \pi_1\ c)$
- $(A \times B) \to C \simeq A \to (B \to C)$
  with $M = \lambda f.\lambda a.\lambda b.f\ (a, b)$
  and $N = \lambda f.\lambda cf\ (\pi_1\ c)\ (\pi_2\ c)$

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For λ-calculus with products and unit type / cartesian closed categories **Semantic (finite sets) [Sol83]**

$\times$	$A \times (B \times C) \simeq (A \times B) \times C$	$A \times B \simeq B \times A$
$\times$ and $\to$	$(A \times B) \to C \simeq A \to (B \to C)$	$A \to (B \times C) \simeq (A \to B) \times (A \to C)$
1	$1 \times A \simeq A$ $\quad 1 \to A \simeq A$	$A \to 1 \simeq 1$

Reduction to Tarski's High School Algebra Problem:
can one find all equalities involving product, exponential and 1 using only

$$a(bc) = (ab)c \qquad ab = ba$$
$$c^{ab} = (c^b)^a \qquad (bc)^a = b^a c^a$$
$$1a = a \quad a^1 = a \quad 1^a = 1$$

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For λ-calculus with products, unit type and sums / cartesian closed categories with binary coproducts [FDB02]

NOT FINITELY AXIOMATISABLE

Give an <u>infinite</u> family of isomorphisms unobtainable from any <u>finite</u> family

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

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$*$ if X not free in A

With the $! - ?_c$, $! - ?_w$ and $?_c - ?_w$ commutations and reductions

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(?_c) \to \overset{\pi}{\vdash ?A, \Gamma}$$

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Optional	$\forall X\ A \simeq A^{*\dagger}$	$\exists X\ A \simeq A^{*\dagger}$	$1 \simeq \bot^{\ddagger}$	$0 \simeq \top^{\clubsuit}$

$*$ if X not free in A

$\dagger$ if $\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \equiv \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists)$

when π is "witness irrelevant"

$\ddagger$ if $\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash}\,(mix_0)}{\vdash \Gamma}\,(mix_2) \equiv \overset{\pi}{\vdash \Gamma}$

$\clubsuit$ with $\dfrac{}{\vdash \Gamma}\,(\emptyset)$

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$*$ if X not free in A

Semantic method complicated:

$\top \otimes A \simeq \top \otimes B$ in most models while syntactically $\top \otimes A \not\simeq \top \otimes B$

$\longrightarrow$ syntactic method, proof-nets as equality easier (but none for full LL)

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For Multiplicative Linear Logic / $\star$-autonomous categories

Syntactic (proof-nets) [BD99]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
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(no proof-nets with units: add them with boxes)

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$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

For Multiplicative-Additive Linear Logic / $\star$-autonomous categories with finite products **Syntactic (proof-nets) [DL23]**

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\ !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\ ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

(no proof-nets with units: add them in sequent calculus)

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \quad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \quad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

For Polarized Linear Logic

Semantic (games, forest isomorphisms) [Lau05]

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \perp \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \& B) \simeq !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq ?A \parr ?B$ $?0 \simeq \perp$	
Quantifiers	$\forall X(A \& B) \simeq \forall X\ A \& \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
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$*$ if X not free in A

Natural perspectives (using proof-nets):

- MELL
- M(A)LL with quantifiers

Retractions in Linear Logic

Retraction $A \trianglelefteq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\rho}{B \vdash A} \quad \overset{\pi}{A \vdash B}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\pi}{A \vdash B} \quad \overset{\rho}{B \vdash A}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Example

$$A \trianglelefteq (A \multimap A) \otimes A$$

No conjecture, even in MLL!

Cantor-Bernstein-Schröeder property

$$A \trianglelefteq B \text{ and } B \trianglelefteq A \implies A \simeq B$$

This property does not always hold!

Retractions & Provability

Lemma

$$!X \trianglelefteq !X \otimes !(X \otimes A) \iff \vdash A \textit{ is provable}$$

$$X \trianglelefteq X \& (X \otimes A) \iff \vdash A \textit{ is provable}$$

$$A \trianglelefteq A \oplus B \iff \vdash B^{\perp}, A \textit{ is provable}$$

Sub-system	Provability: is Γ provable?	Retraction: does $A \trianglelefteq B$?
LL	Undecidable	Undecidable
MELL	TOWER-hard (decidability is open)	TOWER-hard (undecidable if provability is)
MALL	PSPACE-complete	PSPACE-hard & Decidable
MLL	NP-complete	in NP
ALL	P-complete	at least P

(an overview on the complexity of provability can be found in [Lin95])

Results & Conjectures for Retractions

Known Results – all in MLL [Di24]

- Same retractions (non isomorphisms) with and without the units, and with and without the *mix* rules
- Retraction from an atom (universal supertypes): only

$$A \trianglelefteq (A \multimap A) \otimes A$$

  (but only for the *formulas*, not for the *proofs* themselves!)
- Cantor-Bernstein-Schröeder property
- Without unit, if $A \trianglelefteq B$ then $size(A) \leq size(B)$ with equality iff $A \simeq B$

Results & Conjectures for Retractions

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- Cantor-Bernstein-Schröeder property
- Without unit, if $A \trianglelefteq B$ then $\mathit{size}(A) \leq \mathit{size}(B)$ with equality iff $A \simeq B$

Conjectures

- Cantor-Bernstein-Schröeder for MALL (maybe even LL?)
- Retractions of ALL: only

$$A \trianglelefteq A \,\&\, B \quad \text{if } A \vdash B$$

- Retractions of ELL: only

$$?A \trianglelefteq ??A$$

$$?!A \trianglelefteq ?!?!A$$

Conclusion

- Equality of proofs is complex, even worse with rule commutations not from cut-elimination ...
- ... which makes the study of isomorphisms and retractions hard
- Decades-old conjecture for isomorphisms in LL, still open
- Retractions not well-studied, high complexity bounds outside of the smallest sub-systems
- Proof-nets are a good tool for these problems, but limited in scope

Conclusion

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- Retractions not well-studied, high complexity bounds outside of the smallest sub-systems
- Proof-nets are a good tool for these problems, but limited in scope

Thank you!

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Back-Up: Rule Commutations & Provability

Lemma

$$\dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \equiv \quad \dfrac{\dfrac{}{\vdash !A \otimes \top, \top}\,(\top)}{\vdash !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \textit{ is provable}$$

Proof.

- **If A is provable** ($\iff$ $!A$ is provable)

$$\dfrac{\dfrac{}{\vdash !A \otimes \top_A, \top}\,(\top)}{\vdash !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \equiv \dfrac{\dfrac{\vdash !A \qquad \dfrac{}{\vdash \top_A, \top}\,(\top)}{\vdash !A \otimes \top_A, \top}\,(\otimes)}{\vdash !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \equiv \dfrac{\dfrac{\vdash !A \qquad \dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash !A \otimes \top_A, \top}\,(\otimes)}{\vdash !A \otimes \top_A, \top \oplus \top}\,(\oplus_i)$$

$$\equiv \dfrac{\vdash !A \qquad \dfrac{\dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash \top_A, \top \oplus \top}\,(\oplus_i)}{\vdash !A \otimes \top_A, \top \oplus \top}\,(\otimes) \equiv \dfrac{\vdash !A \qquad \dfrac{}{\vdash \top_A, \top \oplus \top}\,(\top_A)}{\vdash !A \otimes \top_A, \top \oplus \top}\,(\otimes)$$

Back-Up: Rule Commutations & Provability

Lemma

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\ (\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\ (\oplus_1) \quad \equiv \quad \dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\ (\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\ (\oplus_2) \iff A \textit{ is provable}$$

Proof.

♦ **If A is not provable** ($\iff$ $!A$ is not provable)

We can compute the full equivalence class in this case:

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top_A, \top}\ (\top)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\ (\oplus_i) \quad \equiv \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\ (\top) \quad \dfrac{}{\vdash \top_A}\ (\top_A)}{\vdash\ !A \otimes \top_A, \top}\ (\otimes)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\ (\oplus_i) \quad \equiv \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\ (\top)}{\vdash\ !A, \top \oplus \top}\ (\oplus_i) \quad \dfrac{}{\vdash \top_A}\ (\top_A)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\ (\otimes)$$

Remark we use $!A$ instead of A to prevent commutations in $\dfrac{}{\vdash\ !A, \top}\ (\top)$, as $!$ is the sole rule not commuting with $\top$

□

Interlude: What about Classical Logic?

Cut-elimination equalizes **all proofs** of a same sequent (by Lafont in [GLT89, Appendix B.1]) $\implies$ $=_\beta$ is the largest possible!

$$\dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash A}}{\vdash C, A}\,(w) \qquad \dfrac{\overset{\rho}{\vdash A}}{C \vdash A}\,(w)}{\vdash A, A}\,(cut)}{\vdash A}\,(c)$$

$$\swarrow_{\beta} \qquad\qquad\qquad\qquad \searrow_{\beta}$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A}}{\vdash A, A}\,(w)}{\vdash A}\,(c) \qquad\qquad\qquad\qquad \dfrac{\dfrac{\overset{\rho}{\vdash A}}{\vdash A, A}\,(w)}{\vdash A}\,(c)$$

$$\equiv \qquad\qquad\qquad\qquad\qquad\qquad \equiv$$

$$\overset{\pi}{\vdash A} \qquad\qquad\qquad\qquad\qquad\qquad \overset{\rho}{\vdash A}$$

Interlude: What about Classical Logic?

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

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Reminder

Cut-elimination equalizes all proofs of a same sequent

Fact

$$A \simeq B \iff A \dashv\vdash B$$

Not very exiting, but **not trivial**:
deciding $A \dashv\vdash B$ is equivalent to deciding provability!