Splitting Idempotents via Interpolation in Multiplicative Linear Logic

Rémi Di Guardia

IRIF – Inria, Université Paris Cité, France

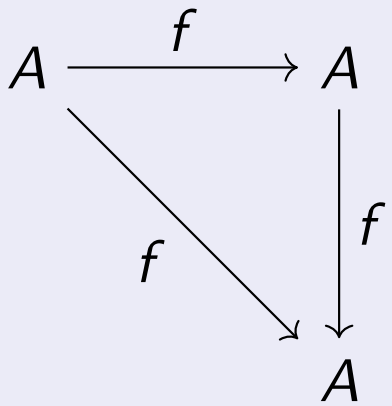
SCALP 2026, Nancy, 21 October 2026





(Split) Idempotents

Definition: Idempotent

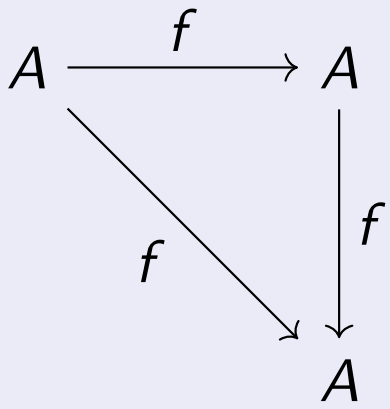


Why studying idempotents?

Gives a better understanding
(*e.g.* in group theory, λ-calculus)

(Split) Idempotents

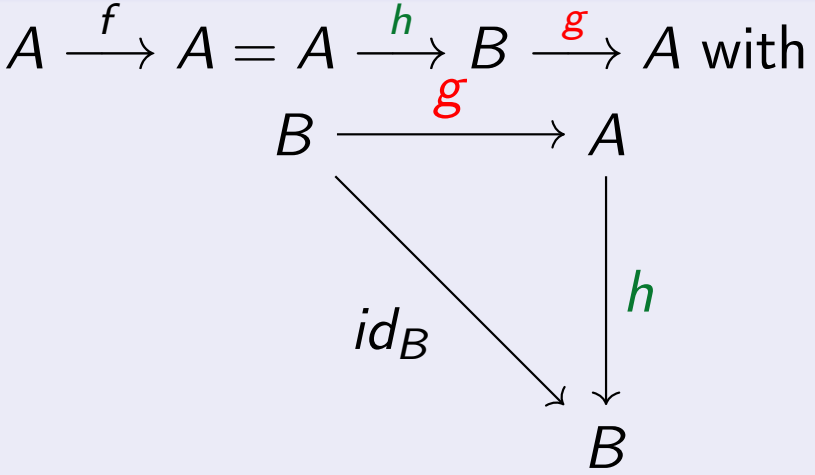
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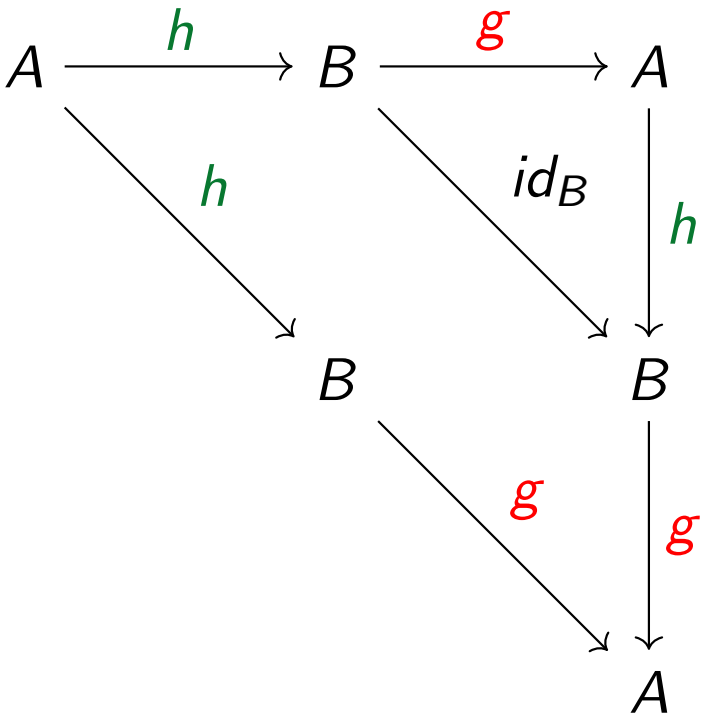


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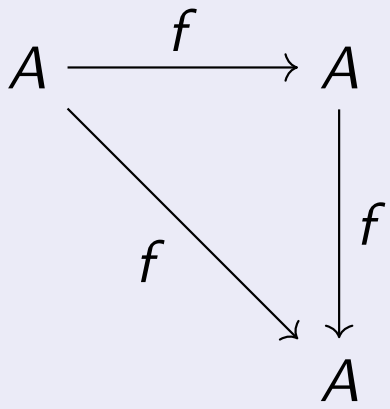
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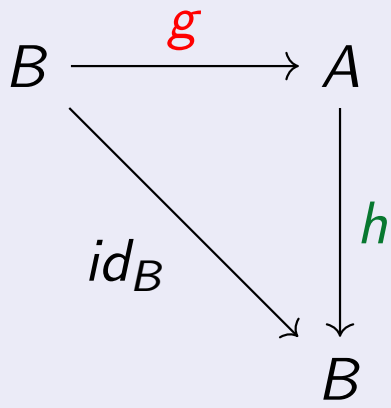
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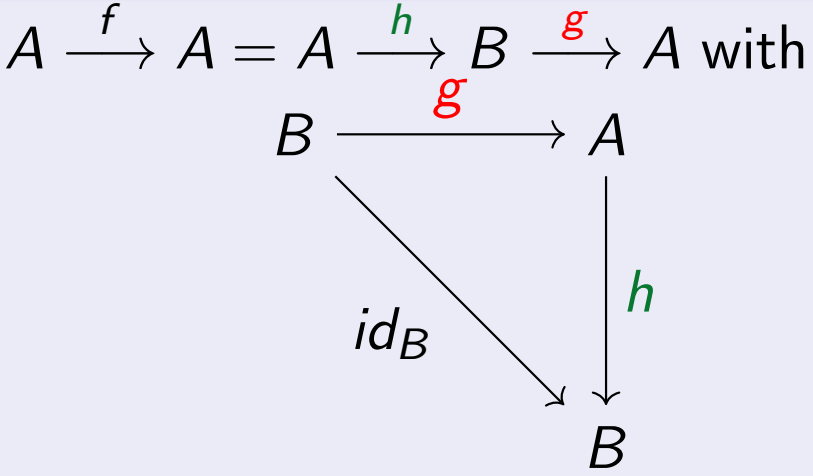
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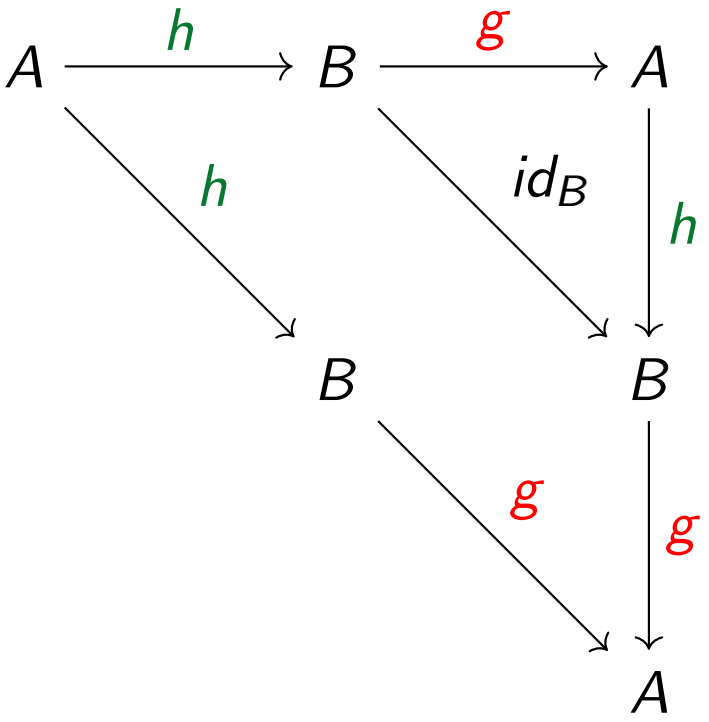
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Definition: Section-Retraction



Definition: Split Idempotent





Idempotent Completeness

Are all idempotents split?

If yes, the category is **idempotent complete**

(*a.k.a.* Karoubi complete, Cauchy complete)

Around idempotent completeness

- Nice property, *e.g.* every absolute (co)limit comes from split idempotents **[Borceux and Dejean 1986]**
- idempotents $\approx$ section-retractions
- Some completion procedures to turn a category idempotent complete

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Set ✓ **[folklore]**

$$\text{idempotent } A \xrightarrow{f} A = A \overset{f^{|}}{\twoheadrightarrow} f(A) \overset{\subseteq}{\hookrightarrow} A$$

$$\text{with } f(A) \overset{\subseteq}{\hookrightarrow} A \overset{f^{|}}{\twoheadrightarrow} f(A) : f(a) \mapsto f(f(a)) = f(a)$$

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MLL/free $\star$-autonomous category ✓ **[this talk]**

Proving MLL is Idempotent Complete

Plan

Given a proof τ of $A \vdash A$ that is idempotent:

1. **Find** a formula B and proofs π of $A \vdash B$ and ϕ of $B \vdash A$ such that

$$\tau = \pi \circ \phi$$

2. **Show** that

$$\phi \circ \pi = id_B$$

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Difficult 1 via **proof-relevant interpolation [Saurin 2025]**

Theorem: Proof-Relevant Interpolation (reformulated)

Let τ be a proof of $A \vdash A$ in linear logic. There exists a formula B and proofs π of $A \vdash B$ and ϕ of $B \vdash A$ such that $\pi \circ \phi = \tau$ and ...

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Multiplicative Linear Logic

Formulas

$$A, B \ ::= \ X \mid X^{\perp} \mid A \parr B \mid A \otimes B \mid \perp \mid 1$$

Involutive Negation

$$(X^{\perp})^{\perp} = X \quad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp} \quad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$$
$$\perp^{\perp} = 1 \quad 1^{\perp} = \perp$$

Rules (one-sided)

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \qquad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \qquad \frac{}{\vdash 1}\ (1)$$

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Axiom-Expansion η **Cut-Elimination β**

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Back to the Plan

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Given a proof τ such that $\dfrac{\overset{\tau}{\vdash A^\perp, A} \quad \overset{\tau}{\vdash A^\perp, A}}{\vdash A^\perp, A}\,(cut) \quad = \quad \tau$

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Plan equality is up to cut-elimination and axiom-expansion

Given a proof τ such that $\dfrac{\overset{\tau}{\vdash A^\perp, A} \quad \overset{\tau}{\vdash A^\perp, A}}{\vdash A^\perp, A}\,(cut) \;=_{\beta\eta}\; \tau$

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Problem

1 is given by interpolation; how to prove 2?

Deciding Equality in MLL

Theorem [Heijltjes and Houston 2016]

MLL proof equivalence is PSPACE-complete.

☹ The general problem is hard ☹ *(the unit $\bot$ moves too freely)*

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Theorem

Two proofs μ and ρ of $\vdash B^{\bot}, B$ are equivalent if and only if

- *their ax-rules regroup the same atoms; and*
- *their* $\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash \bot, 1}\,(\bot)$ *regroup the same* 1 *and* $\bot$*.*

(Linear complexity)

☺ Our particular case is easy ☺ *(each $\bot$ is stuck below a 1)*

Deciding Equality in MLL

Theorem [Heijltjes and Houston 2016]

MLL proof equivalence is PSPACE-complete.

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Theorem

Two (normal) proofs μ and ρ of $\vdash B^{\bot}, B$ are equivalent if and only if (up to isomorphism)

- *their ax-rules regroup the same (occurrences of) atoms; and*
- *their* $\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash \bot, 1}\,(\bot)$ *regroup the same (occurrences of)* 1 *and* $\bot$.

(Linear complexity)

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! Fragile !

Works in MLL but fails when adding additives, exponentials, or quantifiers

New Objective

Updated Plan

Given a proof τ such that $\dfrac{\overset{\tau}{\vdash A^\perp, A} \quad \overset{\tau}{\vdash A^\perp, A}}{\vdash A^\perp, A}\,(cut) \quad =_{\beta\eta} \quad \tau$

1. **Proof-Relevant Interpolation** yields proofs π and ϕ such that

$$\tau \quad =_{\beta\eta} \quad \frac{\overset{\pi}{\vdash A^\perp, B} \quad \overset{\phi}{\vdash B^\perp, A}}{\vdash A^\perp, A}\,(cut)$$

with some properties on π and ϕ: each *ax*-rule uses at least one atom from $A^\perp$ or A, and similarly for units (up to changes in the algorithm)
$\longrightarrow$ obtained by looking at the algorithm = proof, *stronger* than Craig's and Lyndon's interpolation (vocabulary useless here)

2. Given a *normal form* μ of $\dfrac{\overset{\phi}{\vdash B^\perp, A} \quad \overset{\pi}{\vdash A^\perp, B}}{\vdash B^\perp, B}\,(cut)$ **show** it has

ax-rules between dual atoms of $B^\perp$ and B, and similarly for units (= the rules of the identity)

A Matter of Syntax

Syntactic Choices

- Atoms and units behave similarly here
  $\Longrightarrow$ look only at atoms
- Only *ax*-rules matter: forget the rest!

$$\dfrac{\dfrac{\dfrac{}{\vdash X_2^\perp, X_3}\,(ax) \qquad \dfrac{}{\vdash X_1^\perp, X_4}\,(ax)}{\vdash X_1^\perp, X_2^\perp, X_3 \otimes X_4}\,(\otimes)}{\vdash X_1^\perp \parr X_2^\perp, X_3 \otimes X_4}\,(\parr) \quad \rightsquigarrow \quad \dfrac{\dfrac{}{\vdash X_2^\perp, X_3}\,(ax) \qquad \dfrac{}{\vdash X_1^\perp, X_4}\,(ax)}{\vdash X_1^\perp \parr X_2^\perp, X_3 \otimes X_4}$$

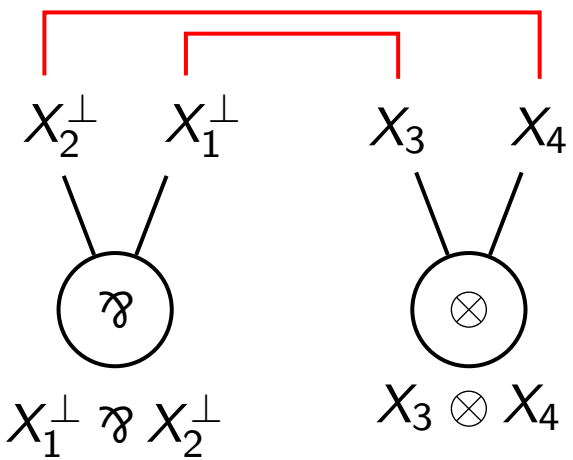
Proof-Tree

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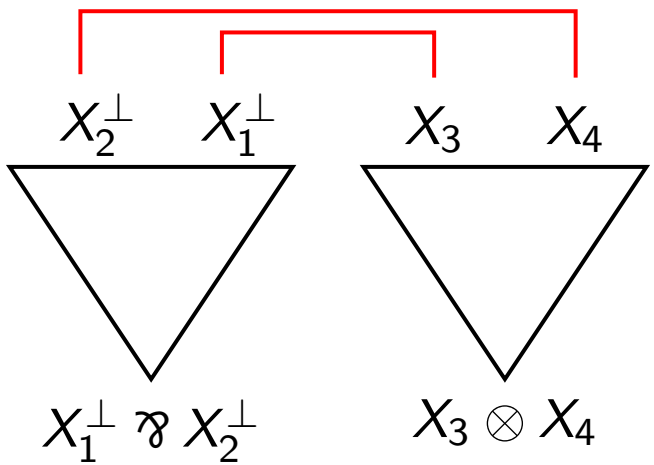
Proof-Tree $\rightsquigarrow$ Formula-trees + Axioms
= Proof-Nets

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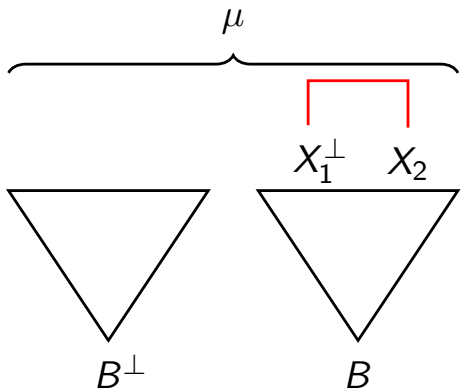
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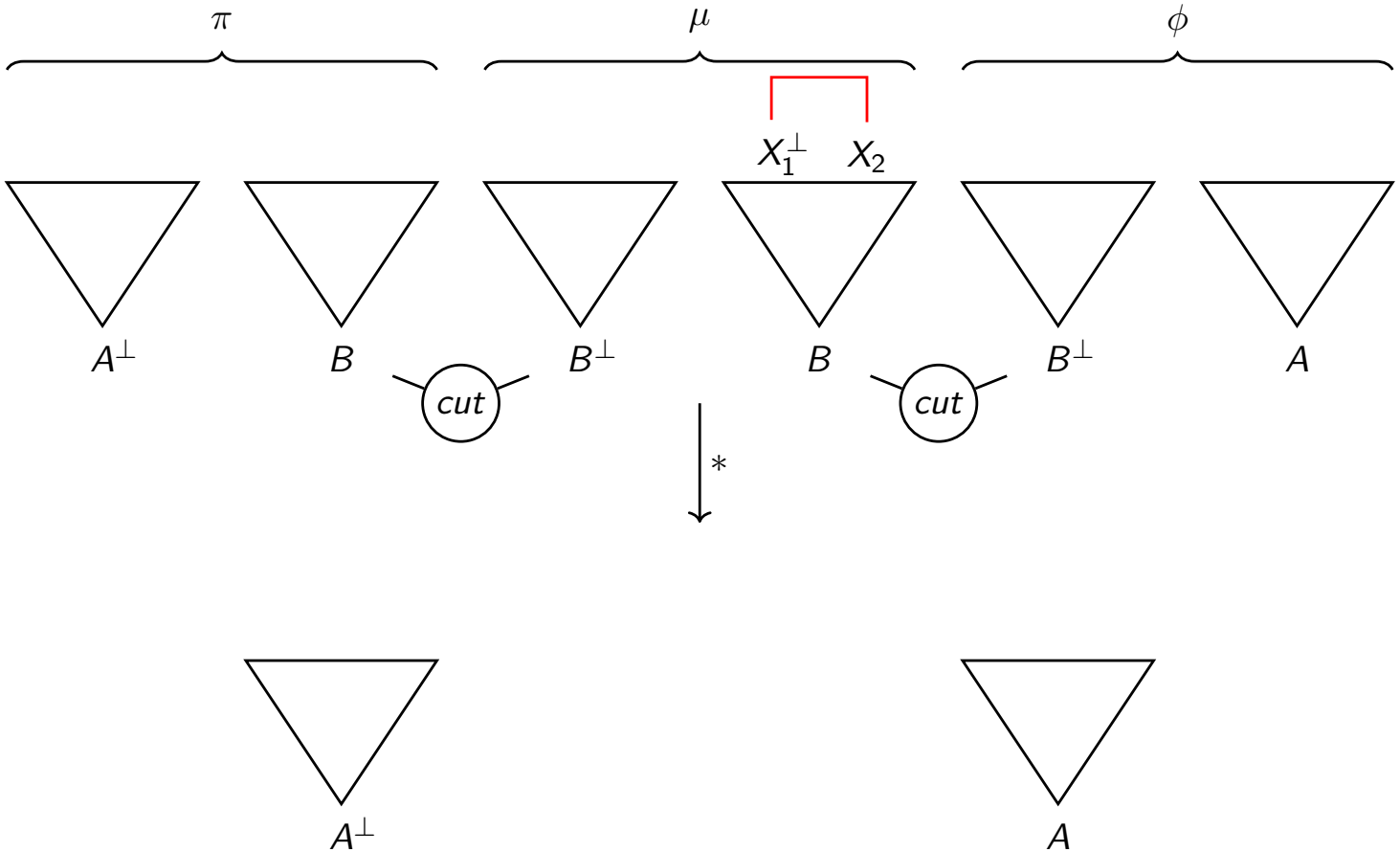


Proof-Tree $\rightsquigarrow$ Formula-trees + Axioms
$=$ Proof-Nets
$\approx$ Geometry Of Interaction
$\approx$ Atomic Flows

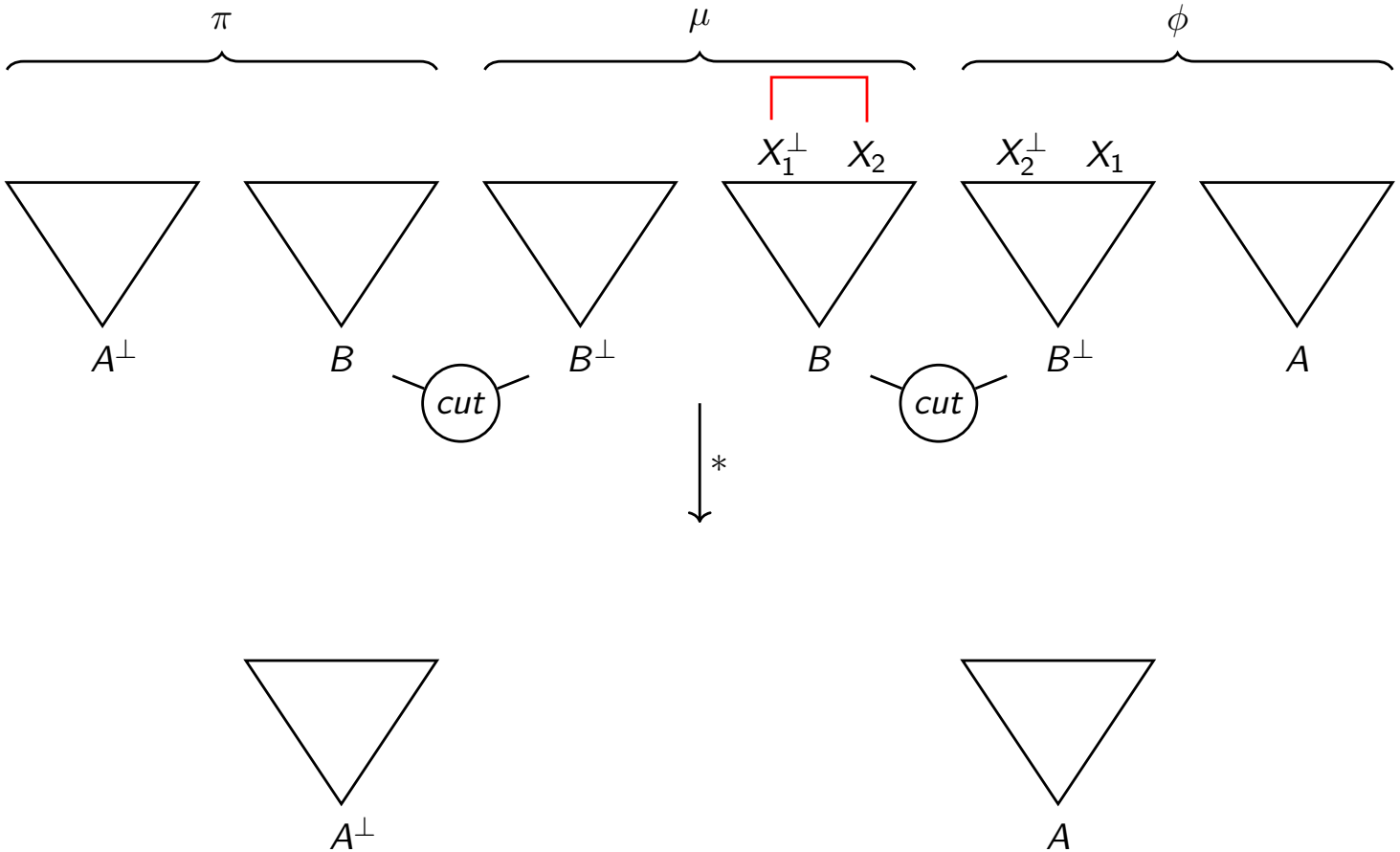
Graphical Axiom-Chasing: No $B - B$ axiom



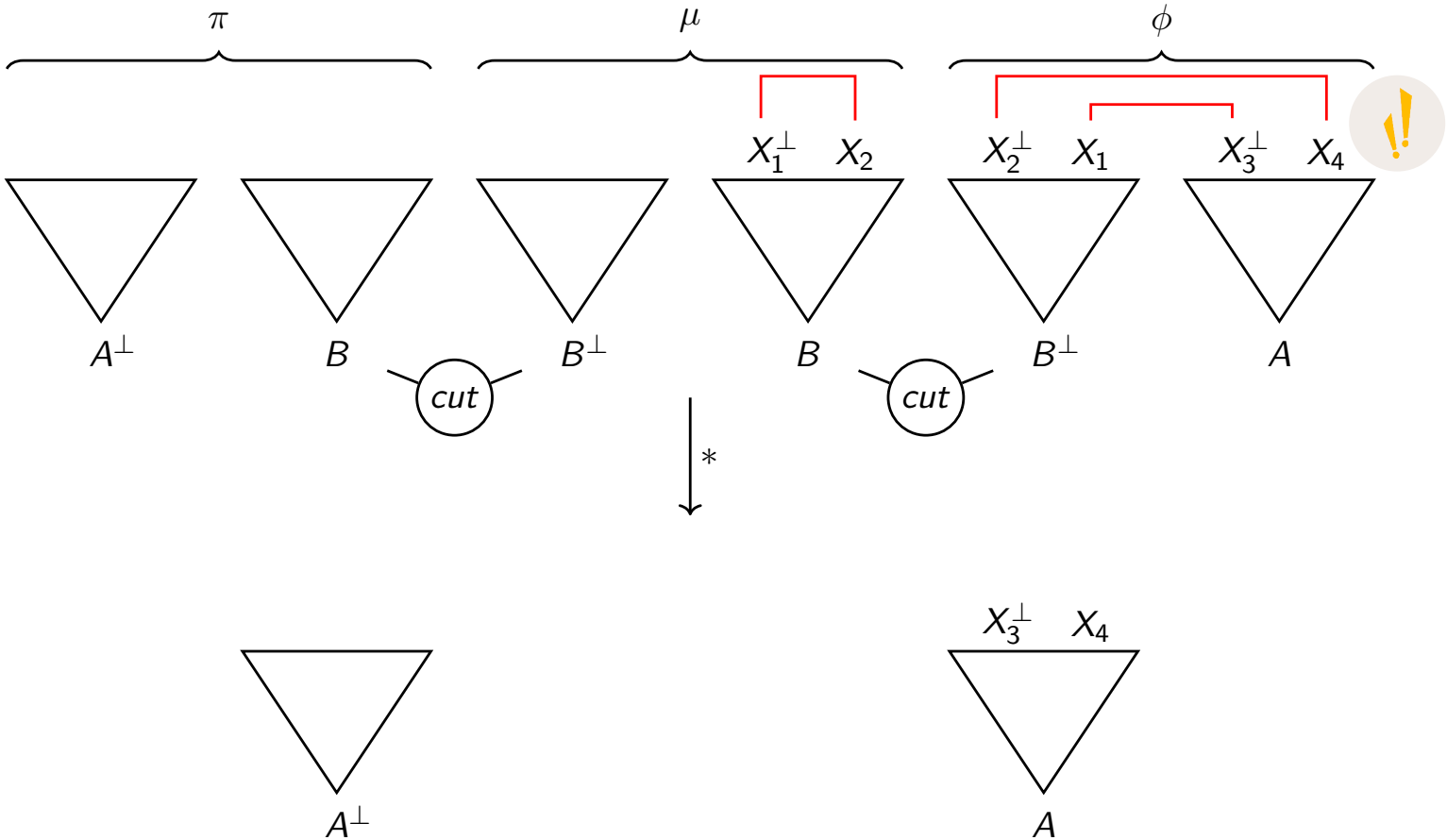
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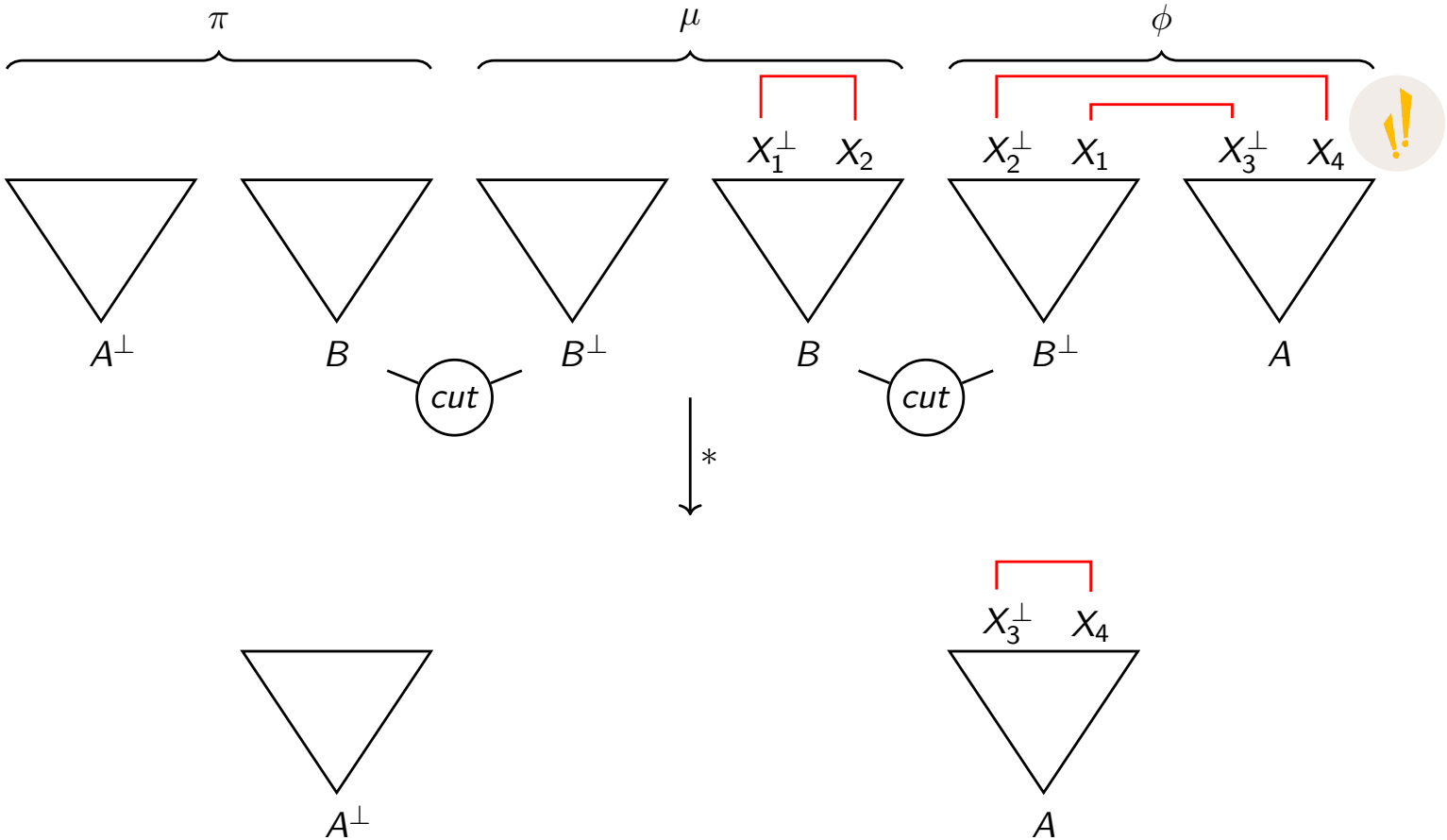
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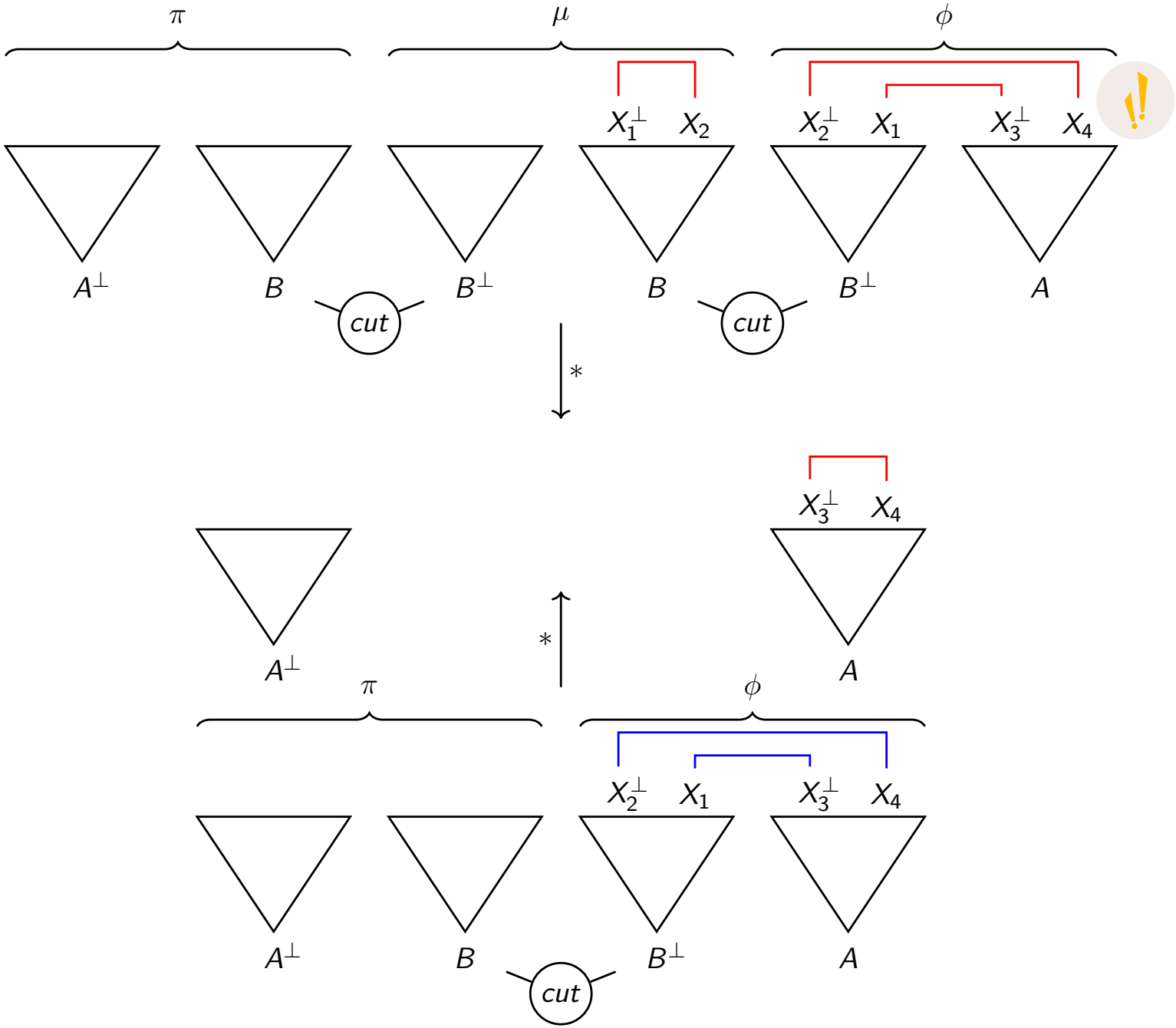
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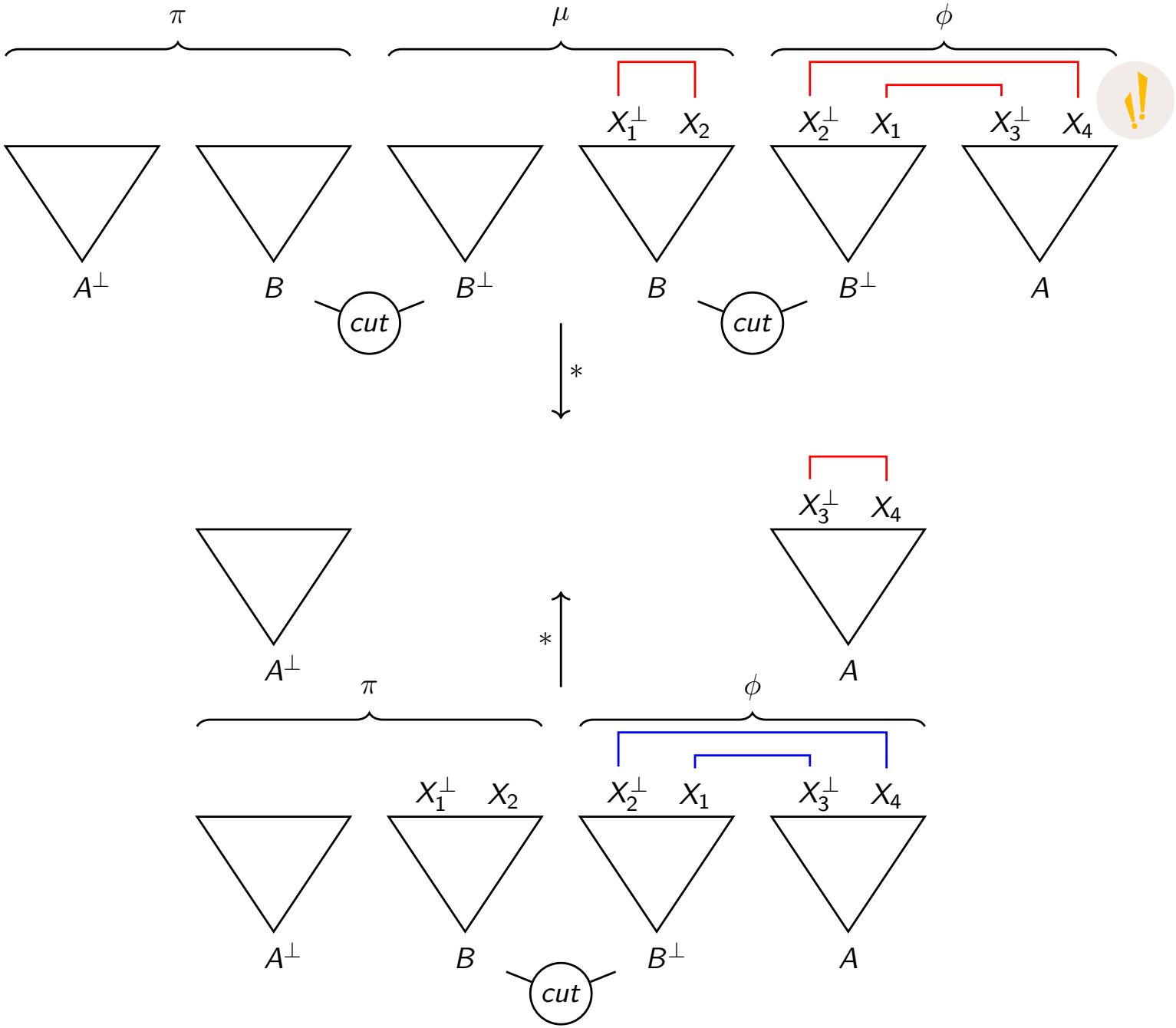
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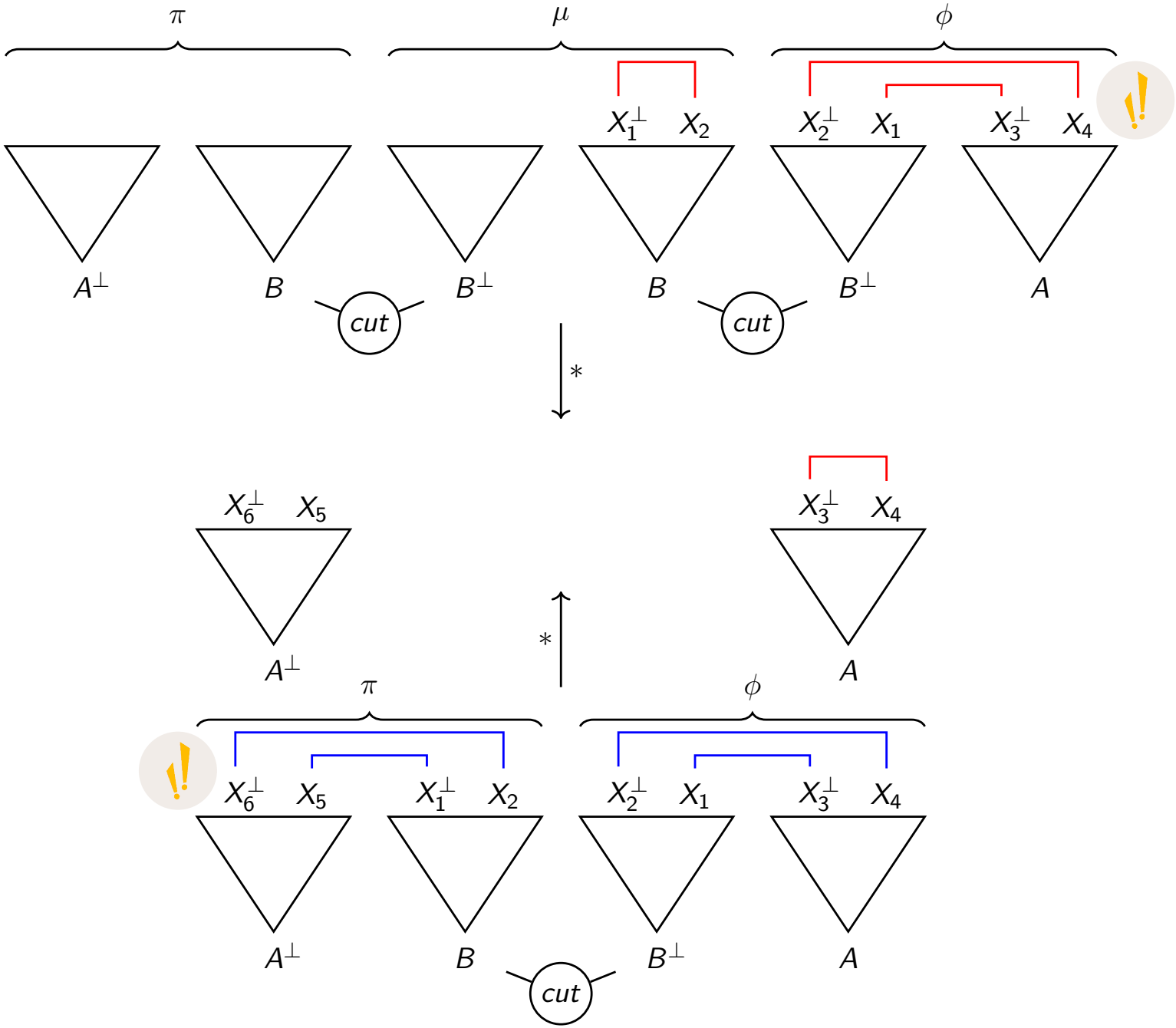
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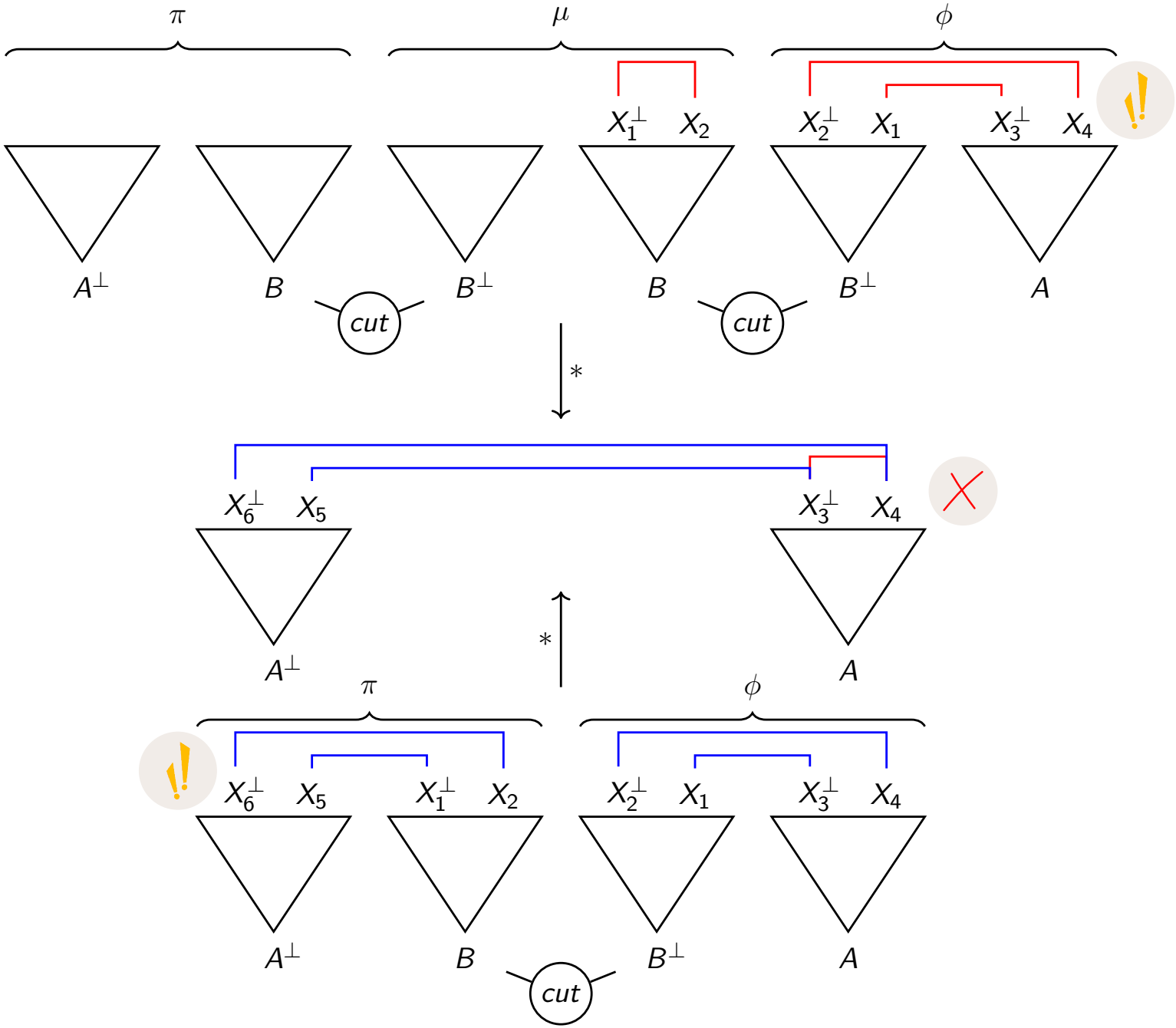
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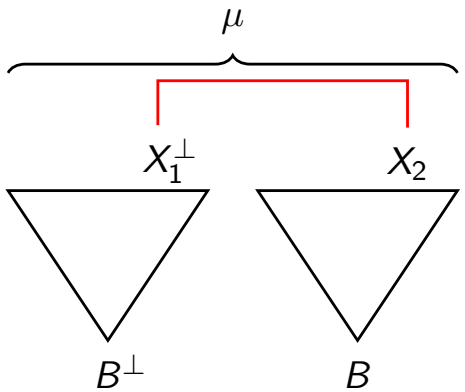
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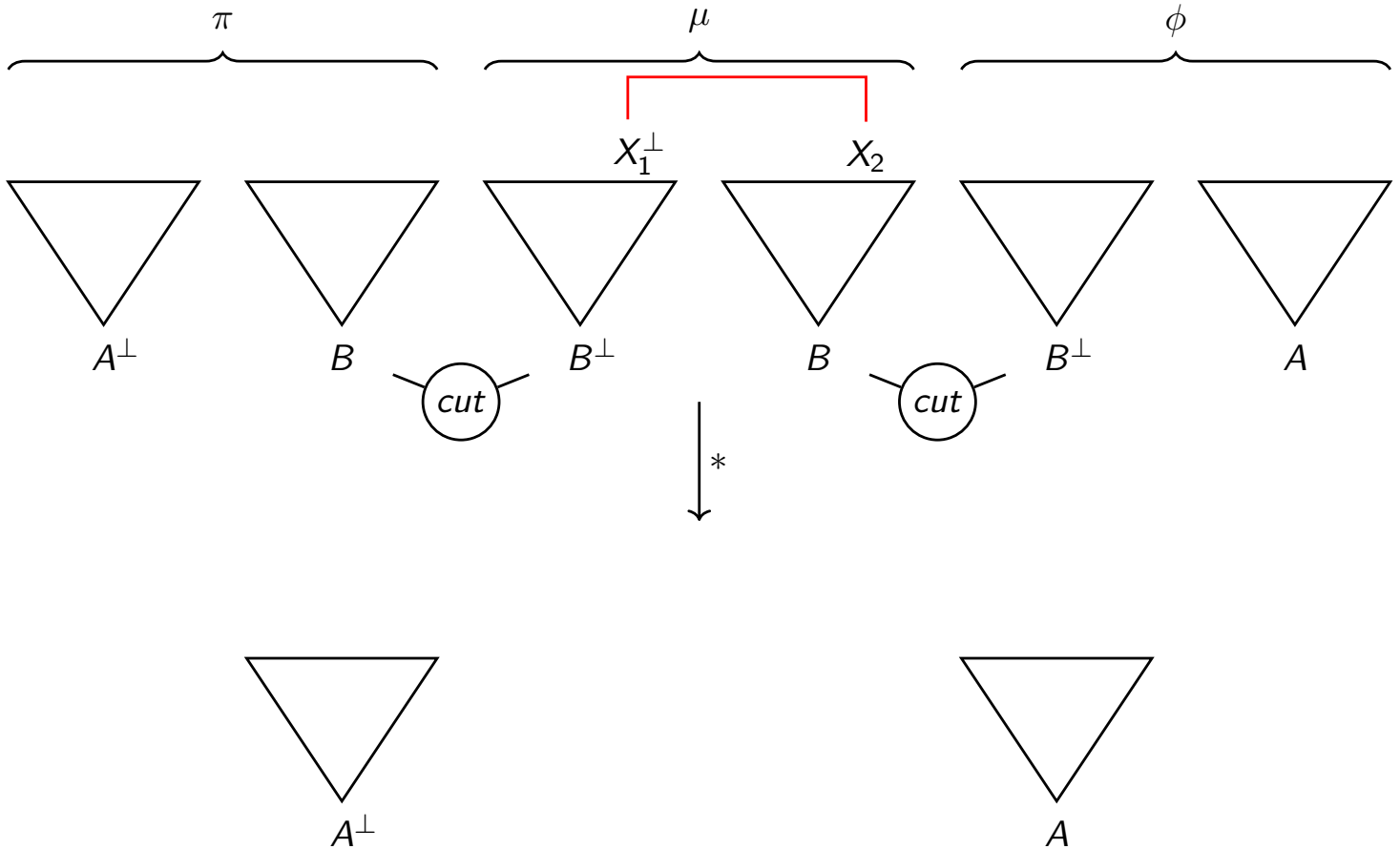
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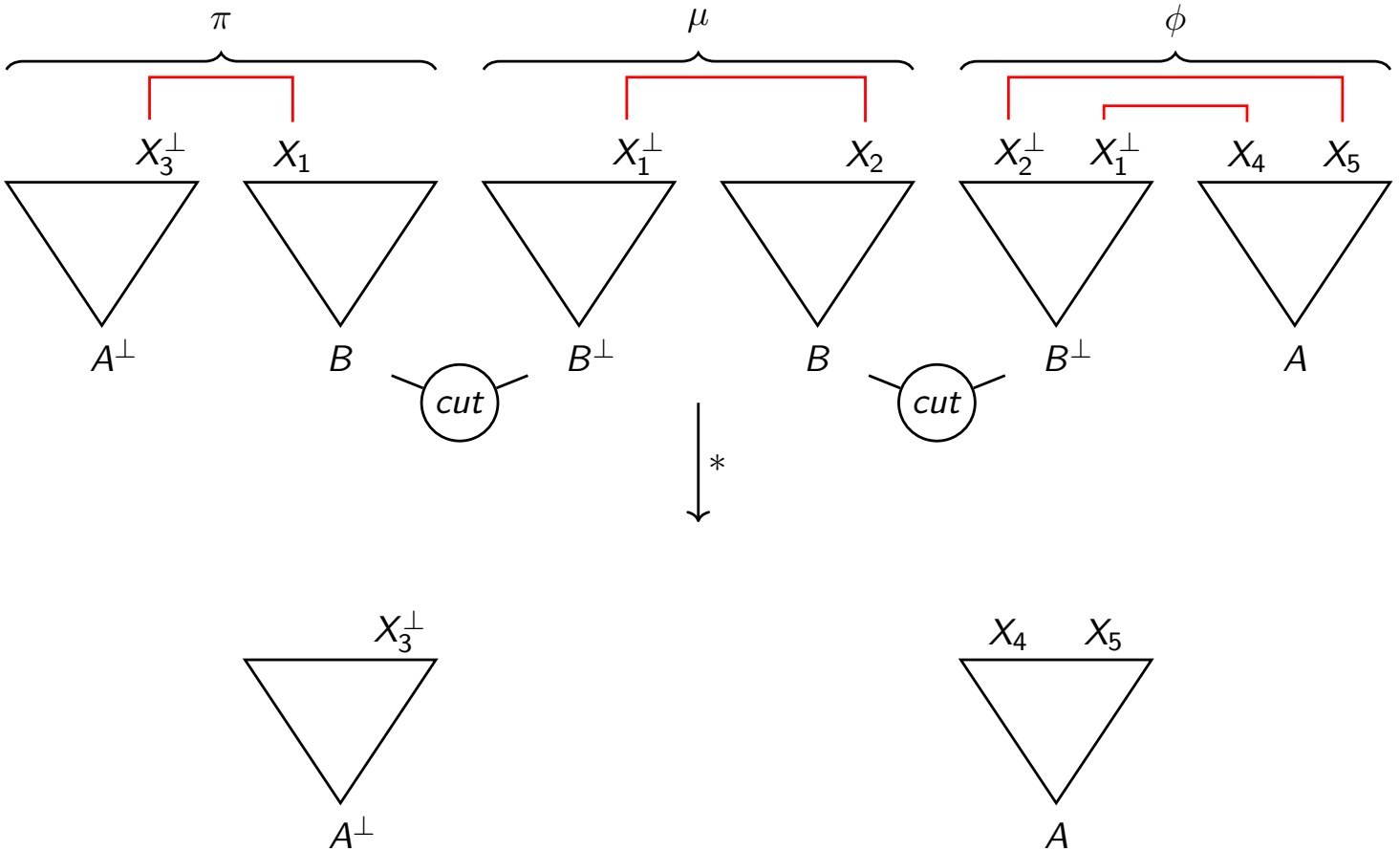
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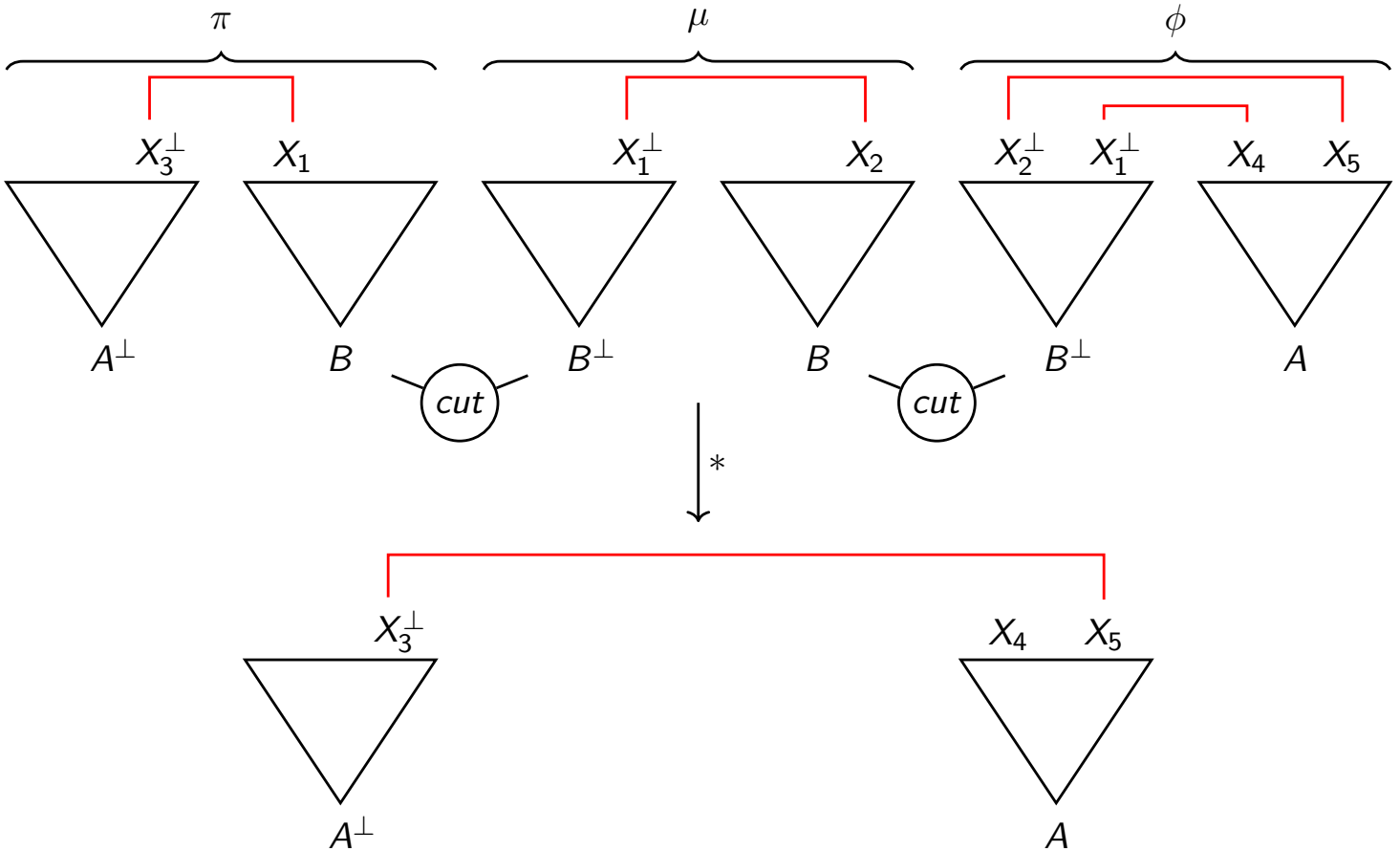
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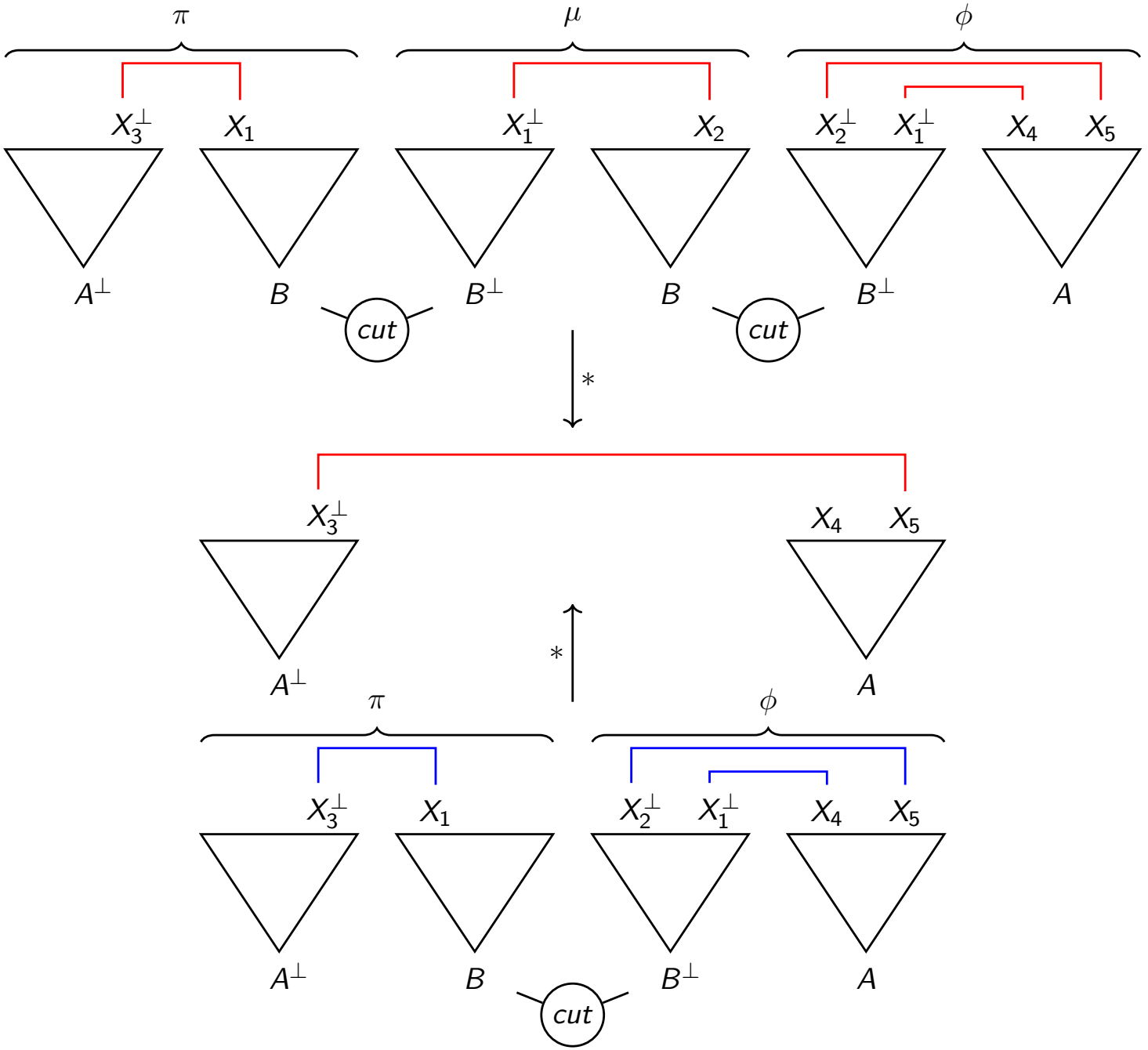
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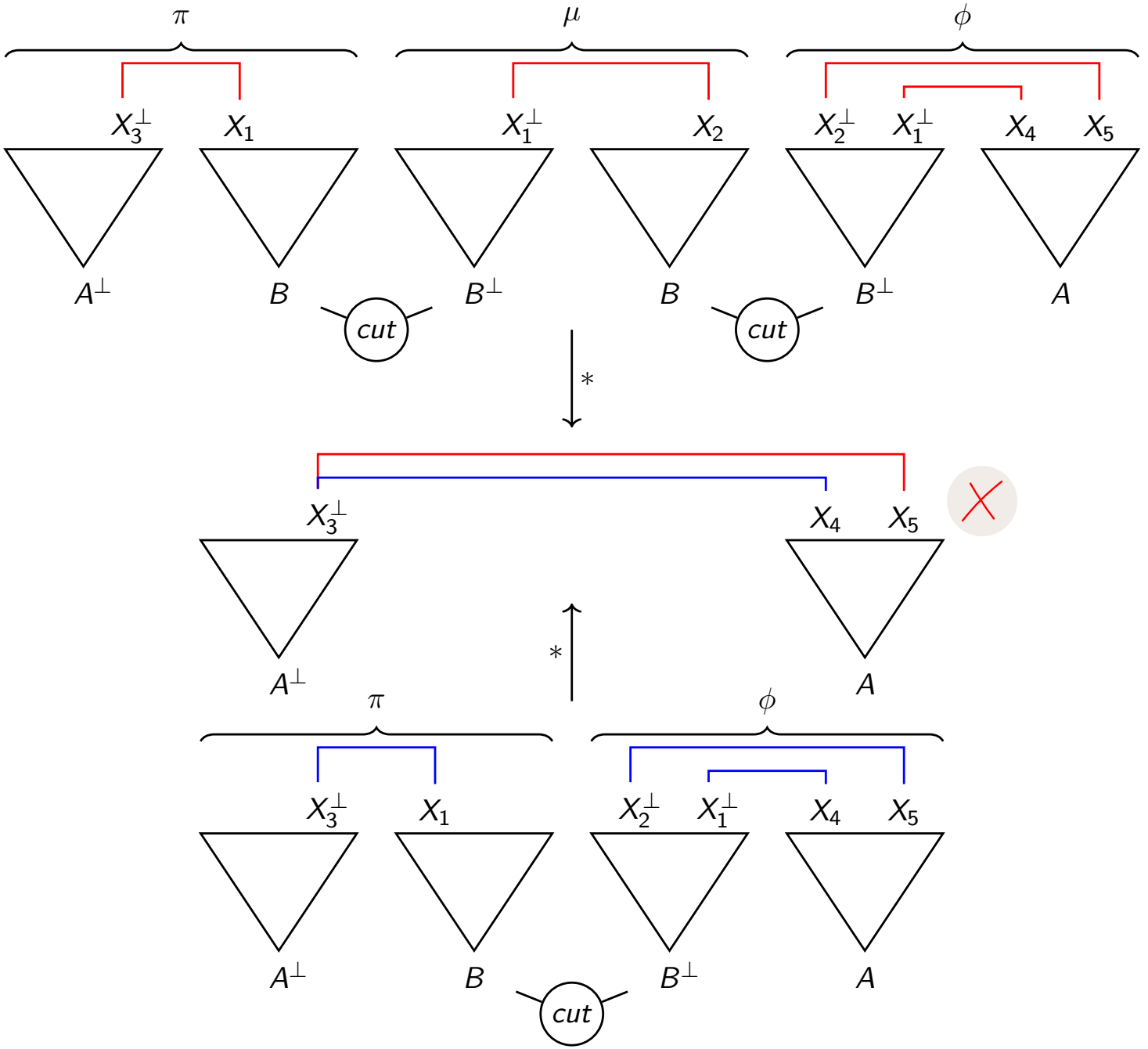
Graphical Axiom-Chasing: Dual axioms



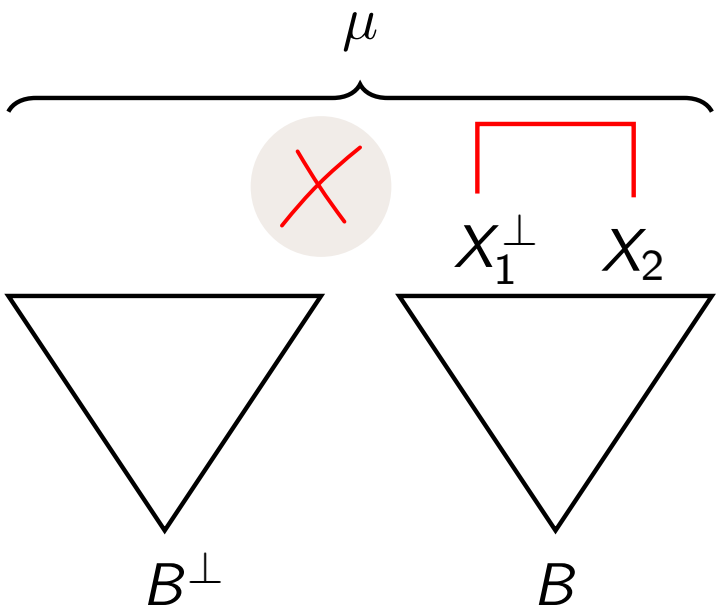
Graphical Axiom-Chasing: Dual axioms

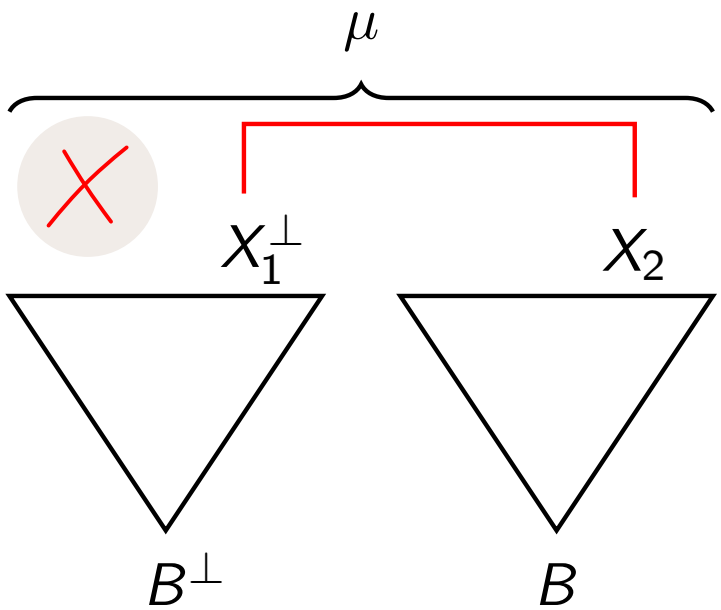


Graphical Axiom-Chasing: Dual axioms



Summing Up





$\Longrightarrow$ Only *ax* between dual atoms in μ $=_{\beta\eta}$ $\dfrac{\overset{\phi}{\vdash B^\perp, A} \qquad \overset{\pi}{\vdash A^\perp, B}}{\vdash B^\perp, B}\ (cut)$

(and the same for units)

$\Longrightarrow$ $\dfrac{\overset{\phi}{\vdash B^\perp, A} \qquad \overset{\pi}{\vdash A^\perp, B}}{\vdash B^\perp, B}\ (cut)$ $=_{\beta\eta}$ $\dfrac{}{\vdash B^\perp, B}\ (ax)$

Conclusion

Theorem

Multiplicative Linear Logic = the free $\star$-autonomous category is idempotent complete.

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Also in presence of (one or both of) the mix-rules.

Proof.

By reduction to the *mix*-free case. □

Conclusion

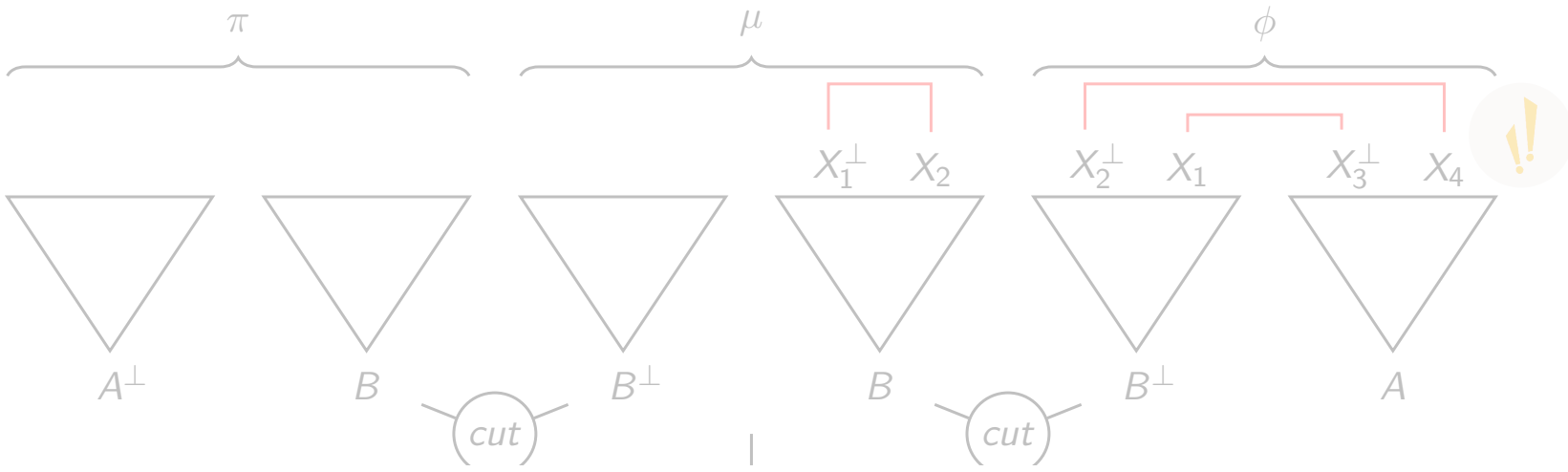
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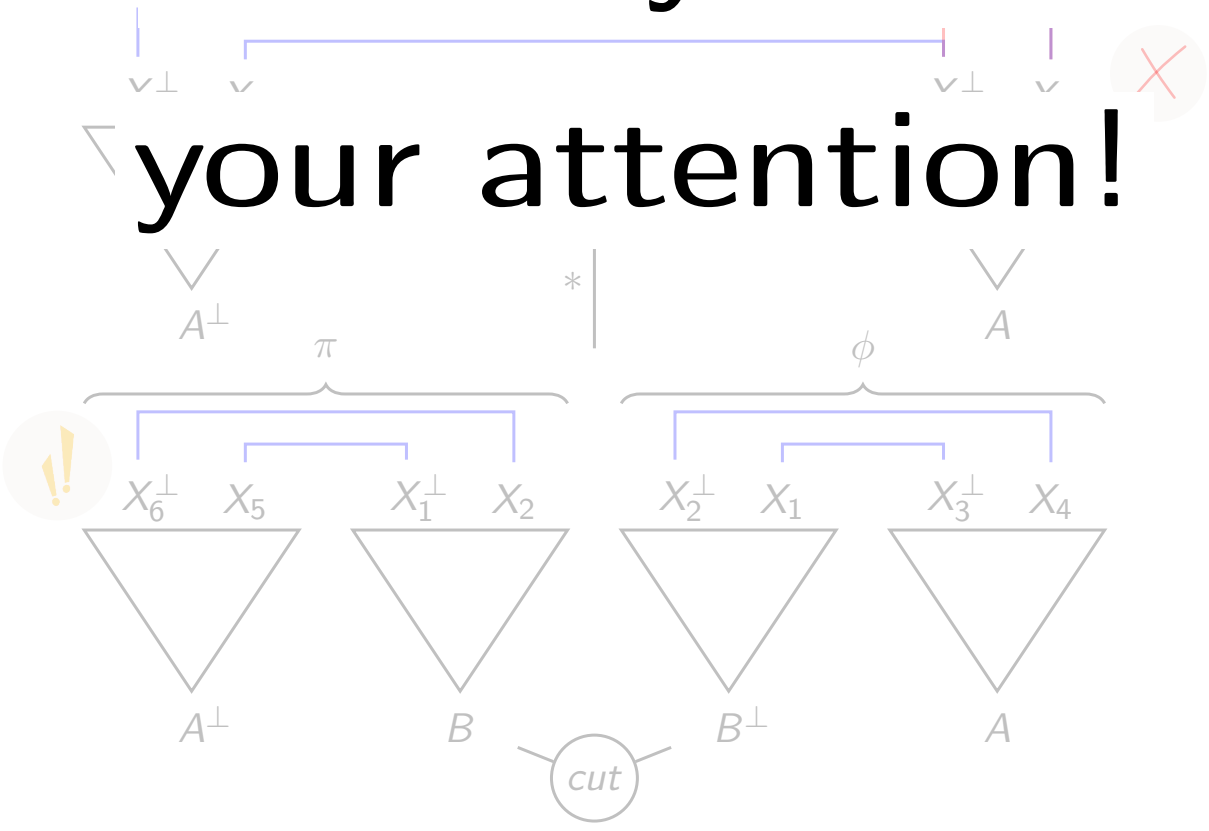
Proof.

By reduction to the *mix*-free case. □

- **Proof-Relevant Interpolation splits idempotents**: quite remarkable
  $\longrightarrow$ something *deeper / more general* behind this?
  just not too many ways to "logically break" a proof in MLL?
  at least points to good properties of the interpolation algorithm
  $\longrightarrow$ other logics where interpolation splits idempotents?
- **Extension** to LL?
  $\longrightarrow$ unit-free MALL seems easy, with units way harder
  $\longrightarrow$ MELL (without *mix*) also harder
  $\longrightarrow$ quantifiers with no unit may be okay
- **Characterization** of idempotents / retractions ?



Thank you for your attention!



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