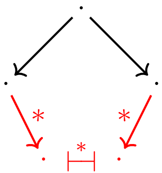
Confluence of Cut-elimination up to Rules commutations in Linear Logic

Rémi Di Guardia

Syntax Meets Semantics, 8 January 2026



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	◇ **Identity** of proofs / terms: *when are two proofs equal?*

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- Equality up to cut-elimination / β-reduction
- Observational equivalence
- Equality up to rules commutations

◇ **Identity** of proofs / terms: *when are two proofs equal?*

◇ Some order does not matter:

$$\dfrac{\dfrac{\vdash A, B, C, D}{\vdash A, B, C \vee D}\,(\vee)}{\vdash A \vee B, C \vee D}\,(\vee) \sim \dfrac{\dfrac{\vdash A, B, C, D}{\vdash A \vee B, C, D}\,(\vee)}{\vdash A \vee B, C \vee D}\,(\vee)$$

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Motivations:

- instance of *when are two morphisms in a category equal?*
- relevant for **isomorphisms**: *when are two formulas A and B equal?*
- useful when looking for a **canonical** representative: proof-nets!

Plan

- **Equality of terms up to β-reduction in λ-calculus**

- **Equality of proofs up to cut-elimination in Linear Logic**
  - Quick sketch of Linear Logic
  - Why is equality more complicated than in λ-calculus?
  - Linking cut-elimination and rules commutations

Simply typed λ-calculus

Terms

$$M, N \ := \ x \mid \lambda x.M \mid M \ N$$

Types

$$A, B \ := \ O \mid A \rightarrow B$$

β**-reduction**

$$(\lambda x.M) \ N \xrightarrow{\beta} M\{N/x\}$$

η**-expansion**

$$M \xrightarrow{\eta} (\lambda x.M \ x)$$

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Syntactic equality is usually not enough:

- **Church encoding:** $\underline{n} := \lambda f.\lambda x.\underbrace{f\ f\ \ldots\ f}_{n \text{ times}}\ x$

  $\underline{2} + \underline{2}$ should be *equivalent* to $\underline{2} + (\underline{1} + \underline{1})$
- **Quotient in category / denotational model:**
  $M =_{\beta\eta} N \implies [\![M]\!] = [\![N]\!]$

$\longrightarrow$ a useful notion of equality is up to computations $=$ $\beta\eta$ equivalence

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<u>Here:</u> ***only equality up to β-reduction*** (the most interesting)

Checking equality of terms

Problem:

- $M =_\beta N$? Give a sequence of terms $M \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} N$
- $M \neq_\beta N$? Prove such a sequence cannot exist!

Checking equality of terms

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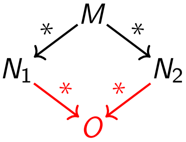
Key results:

- β is **strongly normalizing** (no infinite sequence of reductions)
- β is **confluent**



Corollary

$$M =_\beta N \iff \beta(M) = \beta(N)$$

with $\beta(\cdot)$ the unique normal form of the term

Examples

$$\underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \xleftarrow{\beta^*} \underline{2} + (\underline{1} + \underline{1}) \qquad \underline{2} + \underline{2} \xrightarrow{\beta^*} \underline{4} \neq \underline{3} \xleftarrow{\beta^*} \underline{2} + \underline{1}$$

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Linear Logic

Formulas

$A, B \coloneqq$	$\mid X \mid X^\perp$	*atom*
	$\mid A \parr B \mid A \otimes B \mid \bot \mid 1$	*multiplicative*
	$\mid A \oplus B \mid A \,\&\, B \mid 0 \mid \top$	*additive*
	$\mid ?A \mid !A$	*exponential*
	$\mid \forall X\ A \mid \exists X\ A$	*quantifier*

Involutive Negation / Orthogonality

$$(X^\perp)^\perp = X$$

$$(A \parr B)^\perp = A^\perp \otimes B^\perp \quad (A \otimes B)^\perp = A^\perp \parr B^\perp \quad \bot^\perp = 1 \quad 1^\perp = \bot$$

$$(A \oplus B)^\perp = A^\perp \,\&\, B^\perp \quad (A \,\&\, B)^\perp = A^\perp \oplus B^\perp \quad 0^\perp = \top \quad \top^\perp = 0$$

$$(?A)^\perp = !A^\perp \quad (!A)^\perp = ?A^\perp \qquad (\forall X\ A)^\perp = \exists X\ A^\perp \quad (\exists X\ A)^\perp = \forall X\ A^\perp$$

Sub-systems

- MLL = atom + multiplicative
- MALL = atom + multiplicative + additive
- ...

16 Rules of Linear Logic

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

$$X \text{ not free in } \Gamma \quad \frac{\vdash A, \Gamma}{\vdash \forall X\ A, \Gamma}\ (\forall) \qquad \frac{\vdash A\{B/X\}, \Gamma}{\vdash \exists X\ A, \Gamma}\ (\exists)$$

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Curry-Howard isomorphism: β-reduction $\approx$ cut-elimination

Cut-elimination - 9 Key steps *(Computations)*

ax

$$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \quad \overset{\pi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\,(cut) \quad \xrightarrow{\beta} \quad \overset{\pi}{\vdash A, \Gamma}$$

$\parr - \otimes$

$$\dfrac{\dfrac{\overset{\pi}{\vdash B^\perp, A^\perp, \Gamma}}{\vdash B^\perp \parr A^\perp, \Gamma}\,(\parr) \quad \dfrac{\overset{\rho}{\vdash A, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, \Delta, \Sigma}\,(\otimes)}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\overset{\pi}{\vdash B^\perp, A^\perp, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma}\,(cut) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

$\perp - 1$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \perp, \Gamma}\,(\perp) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash \Gamma}\,(cut) \quad \xrightarrow{\beta} \quad \overset{\pi}{\vdash \Gamma}$$

$\& - \oplus_1$

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$\& - \oplus_2$

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$?d - !$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma}}{\vdash ?A^\perp, \Gamma}\,(?d) \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\overset{\pi}{\vdash A^\perp, \Gamma} \quad \overset{\rho}{\vdash A, ?\Delta}}{\vdash \Gamma, ?\Delta}\,(cut)$$

$?c - !$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp, ?A^\perp, \Gamma}}{\vdash ?A^\perp, \Gamma}\,(?c) \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp, ?A^\perp, \Gamma} \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash ?A^\perp, \Gamma, ?\Delta}\,(cut) \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta, ?\Delta}\,(cut)}{\vdash \Gamma, ?\Delta}\,(?c)$$

$?w - !$

$$\dfrac{\dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash ?A^\perp, \Gamma}\,(?w) \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\overset{\pi}{\vdash \Gamma}}{\vdash \Gamma, ?\Delta}\,(?w)$$

$\forall - \exists$

$$X \text{ not free in } \Gamma \quad \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma}}{\vdash \forall X\ A^\perp, \Gamma}\,(\forall) \quad \dfrac{\overset{\rho}{\vdash A[B/X], \Delta}}{\vdash \exists X\ A, \Delta}\,(\exists)}{\vdash \Gamma, \Delta}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\overset{\pi[B/X]}{\vdash A[B/X]^\perp, \Gamma} \quad \overset{\rho}{\vdash A[B/X], \Delta}}{\vdash \Gamma, \Delta}\,(cut)$$

Cut-elimination - 15 Commutative steps *(To key)*

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B^\perp, \Gamma, \Delta}(cut) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma}(cut) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}(cut)$$

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$$* \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma}}{\vdash A^\perp, \forall X\ B, \Gamma}(\forall) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \forall X\ B, \Gamma, \Delta}(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}(cut)}{* \vdash \forall X\ B, \Gamma, \Delta}(\forall)$$

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$*$ X not free in ...

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$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho}{\vdash C, \Delta}}{\vdash A^\perp, B \otimes C, \Gamma, \Delta}\,(\otimes) \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\tau}{\vdash A, \Sigma}}{\vdash B, \Gamma, \Sigma}\,(cut) \quad \overset{\rho}{\vdash C, \Delta}}{\vdash B \otimes C, \Gamma, \Delta, \Sigma}\,(\otimes)$$

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$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma}}{\vdash A^\perp, \bot, \Gamma}\,(\bot) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \bot, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \bot, \Gamma, \Delta}\,(\bot)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho}{\vdash A^\perp, C, \Gamma}}{\vdash A^\perp, B \,\&\, C, \Gamma}\,(\&) \quad \overset{\tau}{\vdash A, \Delta}}{\vdash B \,\&\, C, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut) \quad \dfrac{\overset{\rho}{\vdash A^\perp, C, \Gamma} \quad \overset{\tau}{\vdash A, \Delta}}{\vdash C, \Gamma, \Delta}\,(cut)}{\vdash B \,\&\, C, \Gamma, \Delta}\,(\&)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma}}{\vdash A^\perp, B \oplus C, \Gamma}\,(\oplus_1) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B \oplus C, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash B \oplus C, \Gamma, \Delta}\,(\oplus_1)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, C, \Gamma}}{\vdash A^\perp, B \oplus C, \Gamma}\,(\oplus_2) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B \oplus C, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, C, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash C, \Gamma, \Delta}\,(cut)}{\vdash B \oplus C, \Gamma, \Delta}\,(\oplus_2)$$

$$\dfrac{\dfrac{}{\vdash A^\perp, \top, \Gamma}\,(\top) \quad \overset{\pi}{\vdash A, \Delta}}{\vdash \top, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{}{\vdash \top, \Gamma, \Delta}\,(\top)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma}}{\vdash A^\perp, ?B, \Gamma}\,(?d) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?d)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, ?B, ?B, \Gamma}}{\vdash A^\perp, ?B, \Gamma}\,(?c) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, ?B, ?B, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash ?B, ?B, \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?c)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma}}{\vdash A^\perp, ?B, \Gamma}\,(?w) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash ?B, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta}\,(cut)}{\vdash ?B, \Gamma, \Delta}\,(?w)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp, B, ?\Gamma}}{\vdash ?A^\perp, !B, ?\Gamma}\,(!) \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash !B, ?\Gamma, ?\Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash ?A^\perp, B, ?\Gamma} \quad \dfrac{\overset{\rho}{\vdash A, ?\Delta}}{\vdash !A, ?\Delta}\,(!)}{\vdash B, ?\Gamma, ?\Delta}\,(cut)}{\vdash !B, ?\Gamma, ?\Delta}\,(!)$$

$$* \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma}}{\vdash A^\perp, \forall X\ B, \Gamma}\,(\forall) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \forall X\ B, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B, \Gamma, \Delta}\,(cut)}{* \vdash \forall X\ B, \Gamma, \Delta}\,(\forall)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B[C/X], \Gamma}}{\vdash A^\perp, \exists X\ B, \Gamma}\,(\exists) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \exists X\ B, \Gamma, \Delta}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B[C/X], \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B[C/X]\Gamma, \Delta}\,(cut)}{\vdash \exists X\ B, \Gamma, \Delta}\,(\exists)$$

$*$ X not free in . . .

Cut-elimination on an example

$$
\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash A^\perp, A \otimes B, B^\perp}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(cut)
$$

Cut-elimination on an example

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash A^\perp, A \otimes B, B^\perp}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(cut)$$

$$\xrightarrow{\beta_{com}}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)}{\vdash A, A^\perp \otimes C^\perp, C}\,(cut) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(\otimes)$$

Cut-elimination on an example

$$\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \dfrac{}{\vdash B, B^{\perp}}\,(ax)}{\vdash A^{\perp}, A \otimes B, B^{\perp}}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A, A^{\perp}}\,(ax) \qquad \dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash A, A^{\perp} \otimes C^{\perp}, C}\,(\otimes)}{\vdash A \otimes B, A^{\perp} \otimes C^{\perp}, B^{\perp}, C}\,(cut)$$

$$\xrightarrow{\beta_{com}}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,(ax) \qquad \dfrac{\dfrac{}{\vdash A, A^{\perp}}\,(ax) \qquad \dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash A, A^{\perp} \otimes C^{\perp}, C}\,(\otimes)}{\vdash A, A^{\perp} \otimes C^{\perp}, C}\,(cut) \qquad \dfrac{}{\vdash B, B^{\perp}}\,(ax)}{\vdash A \otimes B, A^{\perp} \otimes C^{\perp}, B^{\perp}, C}\,(\otimes)$$

$$\xrightarrow{\beta_{key}}$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,(ax) \qquad \dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash A, A^{\perp} \otimes C^{\perp}, C}\,(\otimes) \qquad \dfrac{}{\vdash B, B^{\perp}}\,(ax)}{\vdash A \otimes B, A^{\perp} \otimes C^{\perp}, B^{\perp}, C}\,(\otimes)$$

Plan

- **Equality of terms up to β-reduction in λ-calculus**

- **Equality of proofs up to cut-elimination in Linear Logic**
  - Quick sketch of Linear Logic
  - Why is equality more complicated than in λ-calculus?
  - Linking cut-elimination and rules commutations

Checking equality of proofs

Problem:

- $\pi =_\beta \rho$? Give a sequence of proofs $\pi \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} \rho$
- $\pi \neq_\beta \rho$? Prove such a sequence cannot exist!

Checking equality of proofs

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Can we do the same as in λ-calculus?

- Cut-elimination is **strongly normalizing**?

- Cut-elimination is **confluent**?

Checking equality of proofs

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Can we do the same as in λ-calculus?

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  Almost: an infinite reduction has an infinite suffix made only of

$$\dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \overset{\rho}{\vdash A, \Delta}}{\vdash B^\perp, \Gamma, \Delta}\,(cut) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \quad \overset{\beta}{\longrightarrow} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A^\perp, B^\perp, \Gamma} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A^\perp, \Gamma, \Sigma}\,(cut) \quad \overset{\rho}{\vdash A, \Delta}}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

  $\longrightarrow$ let's just ignore those for now

- Cut-elimination is **confluent**?

Checking equality of proofs

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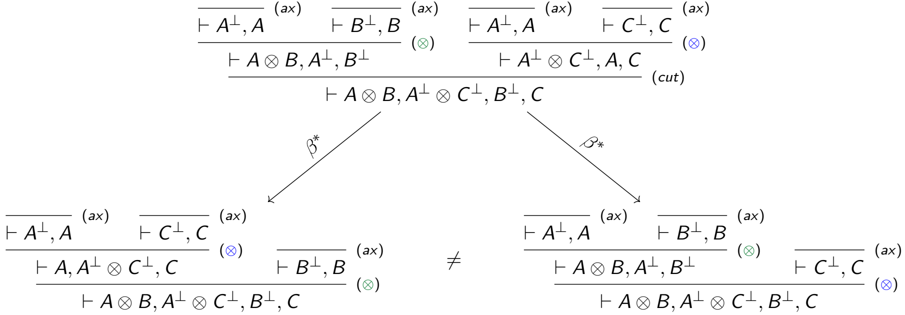
  $\longrightarrow$ let's just ignore those for now

- Cut-elimination is **confluent**?

Not at all!

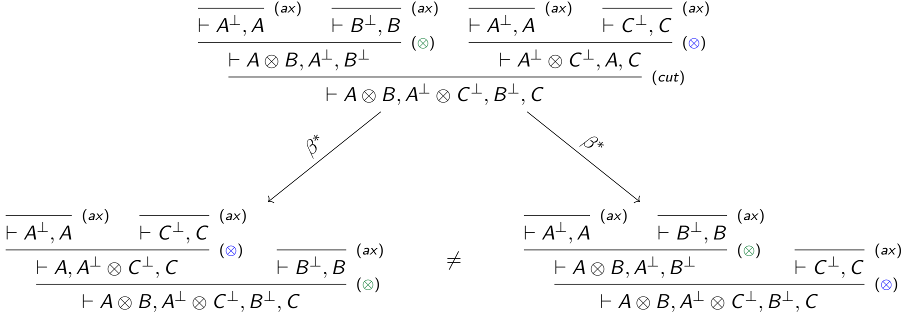
Cut-elimination is not confluent!



Irreversible choice at the beginning:
first commutative case with the left $\otimes$-rule or with the right one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, ...

Cut-elimination is not confluent!



Irreversible choice at the beginning:
first commutative case with the left ⊗-rule or with the right one?

No confluence even in the *simplest* sub-systems: unit-free MLL, ALL, ...

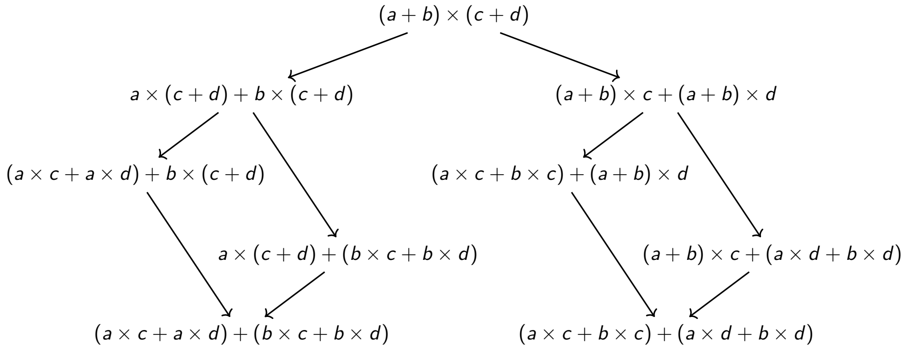
But confluence up to rules commutation!

Intuition: Confluence up to in distributivity

Exercices from junior high school: **distributivity** of $\times$ over $+$

$$a \times (b + c) \to (a \times b) + (a \times c)$$
$$(b + c) \times a \to (b \times a) + (c \times a)$$

Not confluent:



But confluent up to associativity and commutativity of $+$

Rules commutations (from a list of cases)

$$\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \dfrac{\overset{\rho}{\vdash A, D, \Delta} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes) \quad \vdash\!\dashv \quad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \overset{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}\,(\otimes) \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$$

$$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\rho}{\vdash B, C, \Delta} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash B, C \,\&\, D, \Delta}\,(\&)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes) \quad \vdash\!\dashv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\rho}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&)$$

$$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top) \quad \vdash\!\dashv \quad \dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_c)$$

$$\dfrac{}{\vdash \top, A \otimes B, \Gamma, \Delta}\,(\top) \quad \vdash\!\dashv \quad \dfrac{\dfrac{}{\vdash \top, A, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash B, \Delta}}{\vdash \top, A \otimes B, \Gamma, \Delta}\,(\otimes)$$

. . . and many many many more . . .

Rules commutations (from a list of cases)

$$\dfrac{\dfrac{}{\overset{\pi}{\vdash C, \Gamma}} \quad \dfrac{\overset{\rho}{\vdash A, D, \Delta} \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes) \quad \longmapsto \quad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \quad \overset{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}(\otimes) \quad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}(\otimes)$$

$$\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \dfrac{\overset{\rho}{\vdash B, C, \Delta} \quad \overset{\tau}{\vdash B, D, \Delta}}{\vdash B, C \,\&\, D, \Delta}(\&)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}(\otimes) \quad \longmapsto \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\rho}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}(\otimes) \quad \dfrac{\overset{\pi}{\vdash A, \Gamma} \quad \overset{\tau}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}(\&)$$

$$\dfrac{}{\vdash \top, ?A, \Gamma}(\top) \quad \longmapsto \quad \dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}(\top)}{\vdash \top, ?A, \Gamma}(?_c)$$

$$\dfrac{}{\vdash \top, A \otimes B, \Gamma, \Delta}(\top) \quad \longmapsto \quad \dfrac{\dfrac{}{\vdash \top, A, \Gamma}(\top) \quad \overset{\pi}{\vdash B, \Delta}}{\vdash \top, A \otimes B, \Gamma, \Delta}(\otimes)$$

... and many many many more ...

! Non-trivial: duplicates / merges sub-proofs

Rules commutations (from a list of cases)

$$\dfrac{\overset{\pi}{\vdash C,\Gamma} \qquad \dfrac{\overset{\rho}{\vdash A, D, \Delta} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes) \quad\longmapsto\quad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \overset{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}\,(\otimes) \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$$

$$\dfrac{\overset{\pi}{\vdash A,\Gamma} \qquad \dfrac{\overset{\rho}{\vdash B, C, \Delta} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash B, C \,\&\, D, \Delta}\,(\&)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes) \quad\longmapsto\quad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\rho}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&)$$

$$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top) \quad\longmapsto\quad \dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_c)$$

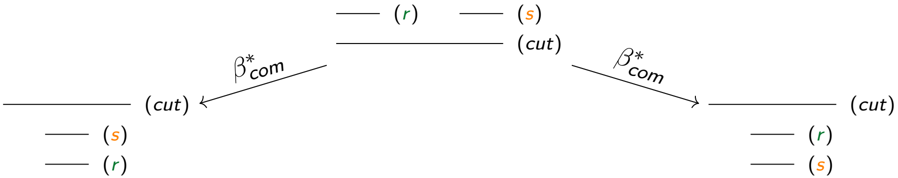
$$\dfrac{}{\vdash \top, A \otimes B, \Gamma, \Delta}\,(\top) \quad\longmapsto\quad \dfrac{\dfrac{}{\vdash \top, A, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash B, \Delta}}{\vdash \top, A \otimes B, \Gamma, \Delta}\,(\otimes)$$

. . . and many many many more . . .

- Non-trivial: duplicates / merges sub-proofs
- Tricky: produces / deletes rules and sub-proofs

Rules commutations (from a general method)

Every pair $\dfrac{\dfrac{}{}\;(s)}{}\;(r) \longmapsto \dfrac{\dfrac{}{}\;(r)}{}\;(s)$ coming from:

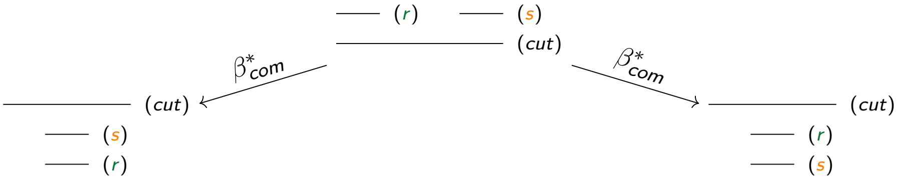


$\approx \#|rules|^2$ commutations $\longrightarrow$ 93 equations in LL!

Rules commutations (from a general method)

Every pair 

coming from:



$\approx \#|rules|^2$ commutations $\longrightarrow$ 93 equations in LL!

Remarks

- $\vdash\!\dashv \;\subseteq\; =_\beta$
- $\vdash\!\dashv$ is the usual (cut-free) commutations without $! - ?_c$ and $! - ?_w$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad \not\vdash\!\dashv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \quad \text{and} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad \not\vdash\!\dashv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

Plan

- **Equality of terms up to β-reduction in λ-calculus**
- **Equality of proofs up to cut-elimination in Linear Logic**
  - Quick sketch of Linear Logic
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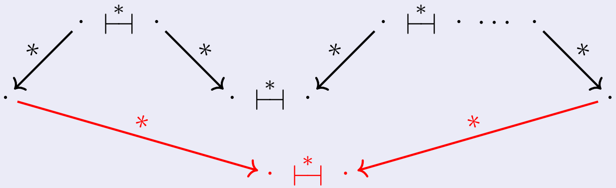
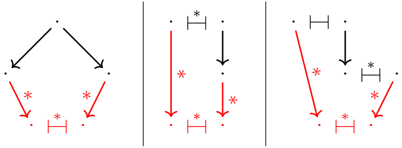
Proving Confluence up to

Definition: Church-Rosser modulo

$\rightarrow$ is **Church-Rosser modulo** an equivalence relation $\stackrel{*}{\vdash\!\dashv}$ when:



How to prove it? Several theorems in rewriting theory. Usual hypotheses:
strong normalization of $\rightarrow \cdot \stackrel{*}{\vdash\!\dashv}$ & **closing some diagrams**



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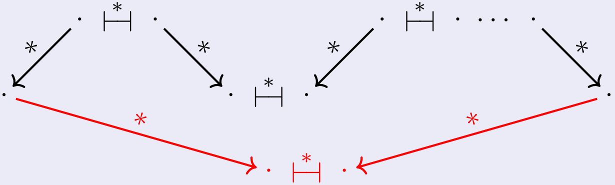

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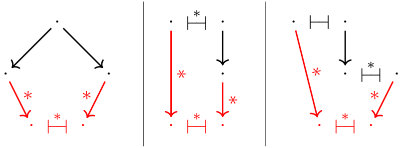

Difficulties:

- $\overset{*}{\vdash\!\dashv}$ is too complex, we prefer $\vdash\!\dashv$
- $\rightarrow \cdot \overset{*}{\vdash\!\dashv}$ is not strongly normalizing!

$\rightarrow \cdot \overset{*}{\vdash\!\dashv}$ is not strongly normalizing!

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}(ax)}{\vdash X^\perp \parr X}(\parr)}{\vdash !(X^\perp \parr X)}(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash ?(X^\perp \otimes X), \top}(\top)}{\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top}(?_w)}{\vdash ?(X^\perp \otimes X), \top}(?_c)}{\vdash \top}(cut)
\;\longrightarrow\;
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}(ax)}{\vdash X^\perp \parr X}(\parr)}{\vdash !(X^\perp \parr X)}(!) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}(ax)}{\vdash X^\perp \parr X}(\parr)}{\vdash !(X^\perp \parr X)}(!) \qquad \dfrac{\dfrac{}{\vdash ?(X^\perp \otimes X), \top}(\top)}{\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top}(?_w)}{\vdash ?(X^\perp \otimes X), \top}(cut)}{\vdash \top}(cut)$$

$$\vdash\!\dashv \qquad\qquad \downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}(ax)}{\vdash X^\perp \parr X}(\parr)}{\vdash !(X^\perp \parr X)}(!) \qquad \dfrac{\dfrac{}{\vdash ?(X^\perp \otimes X), ?(X^\perp \otimes X), \top}(\top)}{\vdash ?(X^\perp \otimes X), \top}(?_c)}{\vdash \top}(cut)
\quad \vdash\!\dashv \quad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}(ax)}{\vdash X^\perp \parr X}(\parr)}{\vdash !(X^\perp \parr X)}(!) \qquad \dfrac{}{\vdash ?(X^\perp \otimes X), \top}(\top)}{\vdash \top}(cut)$$

We have an infinity of **key** cut-elimination cases!

Idea

The problem comes from the production of rules / sub-proofs.

Confluence up to rules commutation

Theorem (Proved in MALL [CP05; DL23]; in redaction for LL)

*Cut-elimination is **Church-Rosser modulo** rules commutation.*

Theorem (inspired from Theorem 2.2 in [AT12])

Let $\vdash\!\dashv$, $\rightarrow$ and $\rightsquigarrow$ be relations such that $\vdash\!\dashv$ is symmetric and $\rightsquigarrow \; \subseteq \; \vdash\!\dashv$.
Set $\Rightarrow \; = \; \rightarrow \; \cup \rightsquigarrow$. Suppose:

1. $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow \; \subseteq \; \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$
3. *if $a \vdash\!\dashv b \rightarrow c$ then either:*
   - $a \rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow c$

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Confluence up to rules commutation

Theorem (Proved in MALL [CP05; DL23]; in redaction for LL)

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Theorem (inspired from Theorem 2.2 in [AT12])

Let $\vdash\!\dashv$, $\rightarrow$ *and* $\rightsquigarrow$ *be relations such that* $\vdash\!\dashv$ *is symmetric and* $\rightsquigarrow\ \subseteq\ \vdash\!\dashv$.
Set $\Rightarrow\ =\ \rightarrow\ \cup \rightsquigarrow$. *Suppose:*

1. $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing*
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Confluence up to rules commutation – SN

Proposition

Set $\rightsquigarrow$ the rules commutations without $\top$-commutations in the direction "creating rules", plus the cut-cut step of cut-elimination.
Then $\xrightarrow{\overline{\beta}} \cdot \rightsquigarrow^*$ is strongly normalizing, with $\xrightarrow{\overline{\beta}} = (\xrightarrow{\beta}$ without cut-cut).

Confluence up to rules commutation – SN

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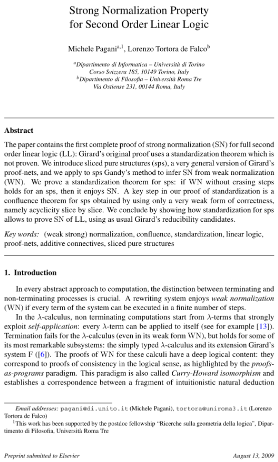

Strong Normalization Property
for Second Order Linear Logic

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Abstract

The paper contains the first complete proof of strong normalization (SN) for full second order linear logic (LL): Girard's original proof uses a standardization theorem which is not proven. We introduce sliced pure structures (sps), a very general version of Girard's proof-nets, and we apply to sps Gandy's method to infer SN from weak normalization (WN). We prove a standardization theorem for sps: if WN without erasing steps holds for an sps, then it enjoys SN. A key step in our proof of standardization is a confluence theorem for sps obtained by using only a very weak form of correctness, namely acyclicity slice by slice. We conclude by showing how standardization for sps allows to prove SN of LL, using as usual Girard's reducibility candidates.

Key words: (weak strong) normalization, confluence, standardization, linear logic, proof-nets, additive connectives, sliced pure structures

1. Introduction

In every abstract approach to computation, the distinction between terminating and non-terminating processes is crucial. A rewriting system enjoys *weak normalization* (WN) if every term of the system can be executed in a finite number of steps.

In the λ-calculus, non terminating computations start from λ-terms that strongly exploit *self-application*: every λ-term can be applied to itself (see for example [13]). Termination fails for the λ-calculus (even in its weak form WN), but holds for some of its most remarkable subsystems: the simply typed λ-calculus and its extension Girard's system F ([6]). The proofs of WN for these calculi have a deep logical content: they correspond to proofs of consistency in the logical sense, as highlighted by the *proofs-as-programs* paradigm. This paradigm is also called *Curry-Howard isomorphism* and establishes a correspondence between a fragment of intuitionistic natural deduction

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[PT10] *almost* do it
"Just" check that **some additions at the start go through** the 61 pages of this technical proof using non-standard proof-nets!

Confluence up to rules commutation

Theorem (Proved in MALL [CP05; DL23]; in redaction for LL)

*Cut-elimination is **Church-Rosser modulo** rules commutation.*

Theorem (inspired from Theorem 2.2 in [AT12])

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Set $\Rightarrow = \rightarrow \cup \rightsquigarrow$. *Suppose:*

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Theorem (Proved in MALL [CP05; DL23]; in redaction for LL)

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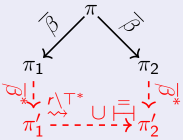
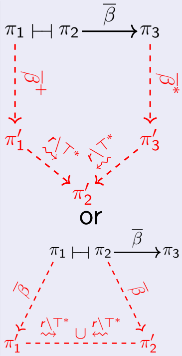
Proof.

#(cut steps)^2
$\approx \#|rules|^2$ cases





#(cut steps)×
#(commutations)
$\approx \#|rules|^3$ cases

□

Confluence up to rules commutation

Theorem (Proved in MALL [CP05; DL23]; in redaction for LL)

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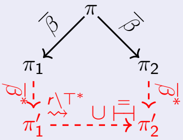

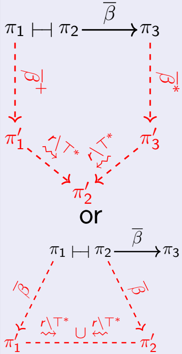

#(cut steps)×
#(commutations)
$\approx$ #|*rules*|^3 cases

□

Thousands of similar cases to check

⟶ horrible and tedious with pen and paper, better in a **proof assistant**!

But the **exchange** rule over-complicates everything...

Confluence up to rules commutation

Theorem (Proved in MALL [CP05; DL23]; in redaction for LL)

*Cut-elimination is **Church-Rosser modulo** rules commutation.*

We still need to add the $cut - cut$ cut-elimination step back.

Proposition

Equality up to cut-elimination = *Equality up to cut-elimination without cut-cut*

Confluence up to rules commutation

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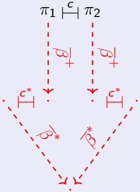
Equality up to cut-elimination	=	*Equality up to cut-elimination without cut-cut*

Proof.

Follows from:

- **strong normalization** of $\xrightarrow{\overline{\beta}} \cdot \overset{c^*}{\vdash\!\dashv}$ with $\overset{c}{\vdash\!\dashv}$ a cut-cut commutation
- 

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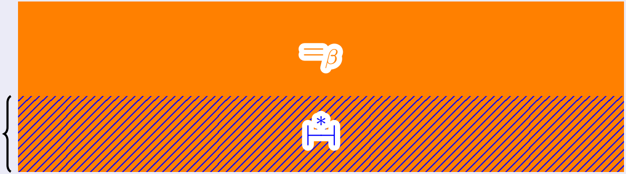
Equality up to cut-elimination	=	*Equality up to cut-elimination without cut-cut*

Corollary: Equality on cut-free proofs

Between cut-free proofs, $=_\beta$ is exactly $=_{\overline{\beta}}$ which is exactly $\overset{*}{\vdash\!\dashv}$.

cut-free proofs

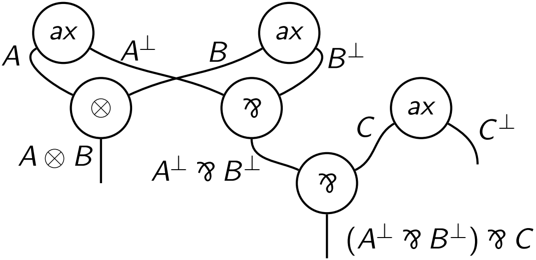


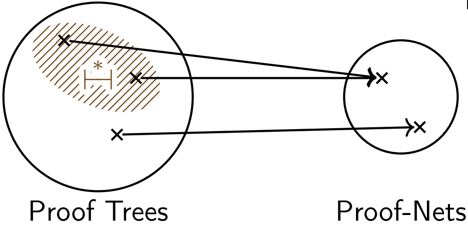
Consequences & Avail

- **"Bureaucracy":** have to <u>order</u> all rules, but some order does not matter and no canonical choice

Consequences & Avail

- **“Bureaucracy”:** have to order all rules, but some order does not matter and no canonical choice
- **Proof-nets:** identify proofs exactly up to rules commutation $\overset{*}{\longmapsto}$
  - $\longmapsto$ is equality of graphs
  - cut-elimination is confluent and has only key steps
  - defined only in some sub-systems of LL





$\overset{*}{\longmapsto}\!\!\dashv$ is better than $=_\beta$ but is not "nice"

Proof Equivalence problem: *given proofs π and ρ, does $\pi \overset{*}{\longmapsto}\!\!\dashv \rho$ hold?*

$\overset{*}{\vdash\dashv}$ is better than $=_\beta$ but is <u>not</u> "nice"

Proof Equivalence problem: *given proofs π and ρ, does $\pi \overset{*}{\vdash\dashv} \rho$ hold?*

Sub-system	Complexity of Proof Equivalence	Method
ALL	in P [Hei11]	through proof-nets
unit-free MLL	in P	through proof-nets
unit-free MALL	in EXPTIME [HG05; HG16]	through proof-nets

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$LL^2 \backslash \top$	decidable	finite number of proofs in a class

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MALL	decidable	finite number of cut-free proofs
$LL^2 \setminus \top$	decidable	finite number of proofs in a class
LL	undecidable	reduces to provability

Lemma

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\ (\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\ (\oplus_1) \quad \overset{*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\ (\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\ (\oplus_2) \iff A \text{ is provable}$$

$\Longrightarrow$ $\overset{*}{\vdash\!\dashv}$ decidable would imply provability decidable, but it is not [Lin95]

Rules commutations & Provability

Lemma

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_{1}) \quad \overset{*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_{2}) \iff A \textit{ is provable}$$

Proof.

♦ **If A is provable** ($\iff$ $!A$ is provable)
We use its proof to find a sequence of commutations:

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top_A, \top}\,(\top)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \vdash\!\dashv \dfrac{\dfrac{\vdash\, !A \qquad \dfrac{}{\vdash \top_A, \top}\,(\top)}{\vdash\, !A \otimes \top_A, \top}\,(\otimes)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \vdash\!\dashv \dfrac{\dfrac{\vdash\, !A \qquad \dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash\, !A \otimes \top_A, \top}\,(\otimes)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i)$$

$$\vdash\!\dashv \dfrac{\vdash\, !A \qquad \dfrac{\dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash \top_A, \top \oplus \top}\,(\oplus_i)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\otimes) \vdash\!\dashv \dfrac{\vdash\, !A \qquad \dfrac{}{\vdash \top_A, \top \oplus \top}\,(\top_A)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\otimes)$$

Rules commutations & Provability

Lemma

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\,(\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{*}{\longmapsto\!\dashv} \quad \dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\,(\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \textit{ is provable}$$

Proof.

♦ **If A is provable** ($\iff$ $!A$ is provable)

♦ **If A is not provable** ($\iff$ $!A$ is not provable)

We can compute the full equivalence class in this case:

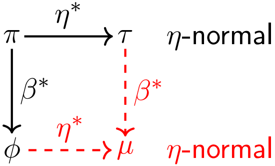
$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top_A, \top}\,(\top)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \quad \longmapsto\!\dashv \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\,(\top) \qquad \dfrac{}{\vdash \top_A}\,(\top_A)}{\vdash\ !A \otimes \top_A, \top}\,(\otimes)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \quad \longmapsto\!\dashv \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\,(\top)}{\vdash\ !A, \top \oplus \top}\,(\oplus_i) \qquad \dfrac{}{\vdash \top_A}\,(\top_A)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\,(\otimes)$$

Remark we use $!A$ instead of A to prevent commutations in $\dfrac{}{\vdash\ !A, \top}\,(\top)$, as $!$ is the sole rule not commuting with $\top$.

□

We may have more than cut-elimination ...

- **Axiom-expansion** and its interactions with cut-elimination



(holds without 2$^{\text{nd}}$ order quantifiers)

We may have more than cut-elimination ...

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad \equiv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad \equiv \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \quad \equiv \quad \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists)$$ when π_B and π_C are "witness irrelevant"

We may have more than cut-elimination . . .

- **Axiom-expansion** and its interactions with cut-elimination
- One may want **more commutations**, yielding even more cases!
- One may want **other rewritings**, with interactions to check

$$\dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(?_c) \quad \rightsquigarrow \quad \overset{\pi}{\vdash ?A, \Gamma}$$

We may have more than cut-elimination . . .

- **Axiom-expansion** and its interactions with cut-elimination
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Thank you!

References I

[AT12] Takahito Aoto and Yoshihito Toyama. "A Reduction-Preserving Completion for Proving Confluence of Non-Terminating Term Rewriting Systems". In: *Logical Methods in Computer Science* 8.1 (Mar. 2012), pp. 1–29. DOI: 10.2168/LMCS-8(1:31)2012.

[CP05] Robin Cockett and Craig Pastro. "A Language For Multiplicative-additive Linear Logic". In: *Electronic Notes in Theoretical Computer Science* 122 (2005). Proceedings of the 10th Conference on Category Theory in Computer Science (CTCS 2004), pp. 23–65. DOI: /10.1016/j.entcs.2004.06.049. URL: https://www.sciencedirect.com/science/article/pii/S1571066105000320.

References II

[DL23] Rémi Di Guardia and Olivier Laurent. “Type Isomorphisms for Multiplicative-Additive Linear Logic”. In: *International Conference on Formal Structures for Computation and Deduction (FSCD)*. Ed. by Marco Gaboardi and Femke van Raamsdonk. Vol. 260. Leibniz International Proceedings in Informatics (LIPIcs). Full version with proofs on `https://hal.science/hal-04082204`. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, July 2023, 26:1–26:21. DOI: `10.4230/LIPIcs.FSCD.2023.26`.

[Hei11] Willem Heijltjes. “Proof Nets for Additive Linear Logic with Units”. In: *Proceedings of the twenty-sixth annual symposium on Logic In Computer Science*. IEEE. Toronto: IEEE Computer Society Press, June 2011, pp. 207–216.

References III

[HG05] Dominic Hughes and Rob van Glabbeek. "Proof Nets for Unit-free Multiplicative-Additive Linear Logic". In: *ACM Transactions on Computational Logic* 6.4 (2005), pp. 784–842. DOI: 10.1145/1094622.1094629.

[HG16] Dominic Hughes and Rob van Glabbeek. *MALL proof nets identify proofs modulo rule commutation*. 2016. URL: https://arxiv.org/abs/1609.04693.

[HH16] Willem Heijltjes and Robin Houston. "Proof equivalence in MLL is PSPACE-complete". In: *Logical Methods in Computer Science* 12.1 (2016). DOI: 10.2168/LMCS-12(1:2)2016.

References IV

[Lin95] Patrick Lincoln. "Deciding provability of linear logic formulas". In: *Advances in Linear Logic*. Ed. by Jean-Yves Girard, Yves Lafont, and Laurent Regnier. Vol. 222. London Mathematical Society Lecture Note Series. Cambridge University Press, 1995, pp. 109–122.

[PT10] Michele Pagani and Lorenzo Tortora de Falco. "Strong normalization property for second order linear logic". In: *Theoretical Computer Science* 411.2 (2010), pp. 410–444.

Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X (A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X (B \parr A)^*$	$\exists X (A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X (B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

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Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \,\&\, (B \,\&\, C) \simeq (A \,\&\, B) \,\&\, C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \,\&\, B \simeq B \,\&\, A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\ !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\ ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

With the $! - ?_c$, $! - ?_w$ and $?_c - ?_w$ commutations and reductions

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\ (!)}{\vdash !A, ?B, ?\Gamma}\ (?_c) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\ (?_c)}{\vdash !A, ?B, ?\Gamma}\ (!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\ (!)}{\vdash !A, ?B, ?\Gamma}\ (?_w) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\ (?_w)}{\vdash !A, ?B, ?\Gamma}\ (!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\ (?_w)}{\vdash ?A, \Gamma}\ (?_c) \to \overset{\pi}{\vdash ?A, \Gamma}$$

Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \& B) \simeq\ !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\ ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \& B) \simeq \forall X\ A \& \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$
Optional	$\forall X\ A \simeq A^{*\dagger}$	$\exists X\ A \simeq A^{*\dagger}$	$1 \simeq \bot^{\ddagger}$	$0 \simeq \top^{\clubsuit}$

$*$ if X not free in A

$\dagger$ if $\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \equiv \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists)$

when π is "witness irrelevant"

$\ddagger$ if $\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash}\,(mix_0)}{\vdash \Gamma}\,(mix_2) \equiv \overset{\pi}{\vdash \Gamma}$

$\clubsuit$ with $\dfrac{}{\vdash \Gamma}\,(\emptyset)$