Confluence Modulo and Undecidability of Cut-Elimination

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Identity of Proofs in Linear Logic

Identity / Equality of proofs: ***when are two proofs π and ρ equal?***

- By Curry-Howard similar to: *when are two λ-terms M and N equal?*
  $\longrightarrow$ equality up to **cut-elimination / β-reduction** $=_\beta$ (at least)
- Another quotient on proofs: **rule commutation** $\overset{r^*}{\longmapsto\!\!\!\!\dashv}$

$$\dfrac{\dfrac{\vdash A, B, C, D}{\vdash A, B, C \parr D}\,(\parr)}{\vdash A \parr B, C \parr D}\,(\parr) \quad \overset{r}{\longmapsto\!\!\!\!\dashv} \quad \dfrac{\dfrac{\vdash A, B, C, D}{\vdash A \parr B, C, D}\,(\parr)}{\vdash A \parr B, C \parr D}\,(\parr)$$

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Motivations:

- ★ Quotient in **categorical/denotational models**
- ★ **Canonical** representative: proof-nets!
- ★ Relevant for **isomorphisms**: *when are two formulas A and B equal?*

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- Another quotient on proofs: **rule commutation** $\overset{r^*}{\longleftrightarrow}$

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Results:

- $=_\beta$ $=$ $\overset{r^*}{\longleftrightarrow}$ in **2$^{\text{nd}}$ order Linear Logic** and most of its fragments *(MALL: [Cockett and Pastro 2005]; [Di Guardia and Laurent 2025])*
- $\overset{r^*}{\longleftrightarrow}$ is **undecidable** in propositional Linear Logic, hence so is $=_\beta$

Plan

- **Confluence of Cut-elimination Modulo Rule Commutation**

$$
\begin{array}{ccc}
 & \pi & \\
\swarrow^{\beta^*} & & \searrow^{\beta^*} \\
\pi_1 & \overset{r^*}{\longmapsto\!\!\dashv} & \pi_2
\end{array}
$$

- **Undecidability of Equality up to Cut / Rule Commutation**

$$\pi =_\beta \rho \text{ ?}$$

$$\pi \overset{r^*}{\longmapsto\!\!\dashv} \rho \text{ ?}$$

How to decide $=_\beta$?

Problematic: $\pi =_\beta \phi$?

Exhibit a sequence $\pi \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} \phi$

or prove such a sequence *cannot exist*!

How to decide $=_\beta$?

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Exhibit a sequence $\pi \xleftarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots \xleftarrow{\beta} \phi$

or prove such a sequence *cannot exist*!

Royal Road: *(e.g. in typed λ-calculus)*

$\mathscr{IF}$ $\xrightarrow{\beta}$ is **strongly normalizing**

no infinite reduction sequence $\pi \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdot \xrightarrow{\beta} \cdots$

$\mathscr{AND}$ $\xrightarrow{\beta}$ is **confluent**

$$\begin{tikzcd}
 & \cdot \arrow[dl, "\beta^*"'] \arrow[dr, "\beta^*"] & \\
\cdot \arrow[dr, red, "\beta^*"'] & & \cdot \arrow[dl, red, "\beta^*"] \\
 & {\color{red}\cdot} &
\end{tikzcd}$$

$\mathscr{THEN}$ $\xrightarrow{\beta}$ has the **unique normal form property**

$\pi =_\beta \phi \iff \beta(\pi) = \beta(\phi)$

Royal Road for $=_\beta$ in Linear Logic?

- $\xrightarrow{\beta}$ is **not** strongly normalizing

$$\dfrac{\dfrac{\vdash A^\perp, B^\perp, \Gamma \quad \vdash A, \Delta}{\vdash B^\perp, \Gamma, \Delta}\,(cut) \qquad \vdash B, \Sigma}{\vdash \Gamma, \Delta, \Sigma}\,(cut) \xrightarrow{\beta} \dfrac{\dfrac{\vdash A^\perp, B^\perp, \Gamma \quad \vdash B, \Sigma}{\vdash A^\perp, \Gamma, \Sigma}\,(cut) \qquad \vdash A, \Delta}{\vdash \Gamma, \Delta, \Sigma}\,(cut)$$

... but infinite reduction sequences have infinite suffixes of such steps!

Let's ignore this cut-elimination step for now

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$$\dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(cut)$$

$$\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes) \quad \xleftarrow{\beta^*} \quad \cdot \quad \xrightarrow{\beta^*} \quad \dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes)$$

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Let's ignore this cut-elimination step for now

- $\xrightarrow{\beta}$ is **not confluent** *but is confluent modulo* $\overset{r^*}{\vdash\!\dashv}$

$$\dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(cut)$$

$$\swarrow_{\beta^+} \qquad\qquad \searrow_{\beta^+}$$

$$\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes)$$

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$$\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{}{}\,(ax) \qquad \dfrac{}{}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax) \quad (\otimes)$$

How to prove it?

$\beta^+ \swarrow \qquad \searrow \beta^+$

$$\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash A^\perp, A \otimes B^\perp, B}\,(\otimes)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes) \qquad \overset{r}{\longmapsto} \qquad \dfrac{\dfrac{\dfrac{}{\vdash C^\perp, C}\,(ax) \qquad \dfrac{}{\vdash A^\perp, A}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A}\,(\otimes) \qquad \dfrac{}{\vdash B^\perp, B}\,(ax)}{\vdash C^\perp, C \otimes A^\perp, A \otimes B^\perp, B}\,(\otimes)$$

Confluence modulo? What's that?

<u>Exercises from junior high school:</u> *distributivity* of $\times$ over $+$

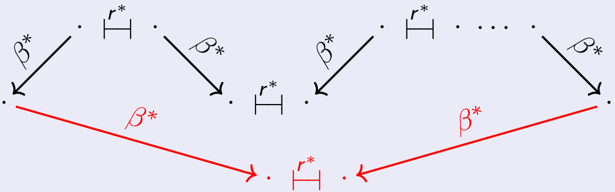
$$a \times (b + c) \to (a \times b) + (a \times c)$$
$$(b + c) \times a \to (b \times a) + (c \times a)$$

Only **confluent** **modulo** associativity and commutativity of $+$

$$\begin{tikzcd}
 & & (a+b)\times(c+d) \arrow[dl] \arrow[drr] & & & \\
 & a\times(c+d)+b\times(c+d) \arrow[dl] \arrow[ddr] & & & (a+b)\times c+(a+b)\times d \arrow[dl] \arrow[ddr] & \\
(a\times c+a\times d)+b\times(c+d) \arrow[ddr] & & & (a\times c+b\times c)+(a+b)\times d \arrow[ddr] & & \\
 & & a\times(c+d)+(b\times c+b\times d) \arrow[dl] & & & (a+b)\times c+(a\times d+b\times d) \arrow[dl] \\
 & (a\times c+a\times d)+(b\times c+b\times d) & & & (a\times c+b\times c)+(a\times d+b\times d) &
\end{tikzcd}$$

Proving Confluence Modulo

Definition: $\xrightarrow{\beta}$ is Church-Rosser modulo $\overset{r^*}{\vdash\!\!\dashv}$



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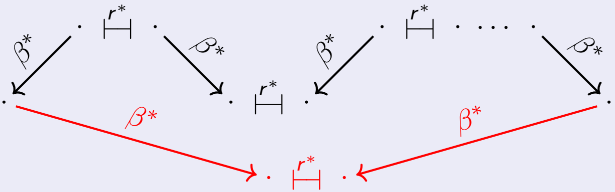
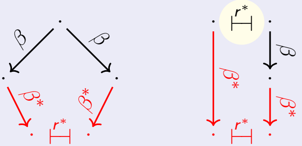


Prototype n°1 of Church-Rosser Modulo Theorem [Huet 1980]

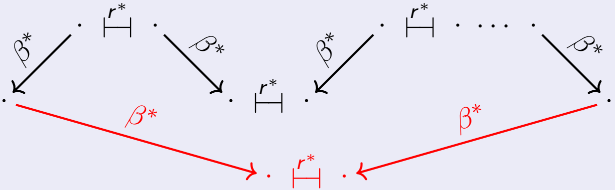
If $\xrightarrow{\beta}$ is strongly normalizing and $\{$*diagrams to close with* $\overset{r^*}{\vdash\!\dashv}\}$
then $\xrightarrow{\beta}$ is ***Church-Rosser modulo*** $\overset{r^*}{\vdash\!\dashv}$



<u>**Problem:**</u> $\overset{r^*}{\vdash\!\dashv}$ is too complex, we prefer $\overset{r}{\vdash\!\dashv}$

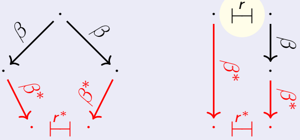
Proving Confluence Modulo

Definition: $\xrightarrow{\beta}$ is Church-Rosser modulo $\overset{r^*}{\vdash\!\dashv}$



Prototype n°2 of Church-Rosser Modulo Theorem [Huet 1980]

If $\xrightarrow{\beta} \cdot \overset{r^*}{\vdash\!\dashv}$ *is strongly normalizing and* $\{$*diagrams to close with* $\overset{r}{\vdash\!\dashv}\}$
then $\xrightarrow{\beta}$ *is **Church-Rosser modulo*** $\overset{r^*}{\vdash\!\dashv}$



<u>Let's try this!</u>

Step 1: Strong Normalization of $\xrightarrow{\beta} \cdot \overset{r^*}{\vdash\!\!\dashv}$

What could possibly go wrong? Let's mix *additive units* and *exponentials*!

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash ?\bot, \top}\,(\top)}{\vdash ?\bot, ?\bot, \top}\,(?_w)}{\vdash ?\bot, \top}\,(?_c)}{\vdash \top}\,(cut)$$

Step 1: Strong Normalization of $\xrightarrow{\beta} \cdot \stackrel{r^*}{\longmapsto}$

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$$
\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash ?\bot, \top}\,(\top)}{\vdash ?\bot, ?\bot, \top}\,(?_w)}{\vdash ?\bot, \top}\,(?_c)}{\vdash \top}\,(\mathit{cut})
\quad \xrightarrow{\quad\beta\quad} \quad
\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{\dfrac{}{\vdash ?\bot, \top}\,(\top)}{\vdash ?\bot, ?\bot, \top}\,(?_w)}{\vdash ?\bot, \top}\,(\mathit{cut})}{\vdash \top}\,(\mathit{cut})
$$

$$
\Big\downarrow \beta
$$

$$
\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash !1}\,(!) \qquad \dfrac{}{\vdash ?\bot, \top}\,(\top)}{\vdash \top}\,(\mathit{cut})
$$

What could possibly go wrong? Let's mix *additive units* and *exponentials*!

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash\, !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash\, ?\bot, \top}\,(\top)}{\vdash\, ?\bot, ?\bot, \top}\,(?_w)}{\vdash\, ?\bot, \top}\,(?_c)}{\vdash \top}\,(cut) \quad \xrightarrow{\beta} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash\, !1}\,(!) \qquad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash\, !1}\,(!) \qquad \dfrac{\dfrac{}{\vdash\, ?\bot, \top}\,(\top)}{\vdash\, ?\bot, ?\bot, \top}\,(?_w)}{\vdash\, ?\bot, \top}\,(cut)}{\vdash \top}\,(cut)$$

$$\Big\downarrow \beta$$

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash\, !1}\,(!) \qquad \dfrac{\dfrac{}{\vdash\, ?\bot, ?\bot, \top}\,(\top)}{\vdash\, ?\bot, \top}\,(?_c)}{\vdash \top}\,(cut) \quad \overset{r}{\vdash\!\!\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1)}{\vdash\, !1}\,(!) \qquad \dfrac{}{\vdash\, ?\bot, \top}\,(\top)}{\vdash \top}\,(cut)$$

Step 1: Strong Normalization of $\xrightarrow{\beta} \cdot \overset{r^*}{\longmapsto}$

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$$\overset{r}{\longmapsto} \qquad \qquad \big\downarrow \beta$$

Infinity of key cut-elimination steps!

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}(1)}{\vdash !1}(!) \qquad \dfrac{\dfrac{}{\vdash ?\bot, ?\bot, \top}(\top)}{\vdash ?\bot, \top}(?_c)}{\vdash \top}(cut) \quad \overset{r}{\longmapsto} \quad \dfrac{\dfrac{\dfrac{}{\vdash 1}(1)}{\vdash !1}(!) \qquad \dfrac{}{\vdash ?\bot, \top}(\top)}{\vdash \top}(cut)$$

Intuition

Problem: production of rules / sub-proofs by commutation with $\top$

Let's be more subtile

Proposition (inspired from [Aoto and Toyama 2012])

Let $\rightarrow$, $\vdash\!\dashv$ and $\rightsquigarrow$ be relations such that $\vdash\!\dashv$ is symmetric and $\rightsquigarrow\ \subseteq \vdash\!\dashv$. Set $\Rightarrow\ =\ \rightarrow\ \cup \rightsquigarrow$. Then $\rightarrow$ is ***Church-Rosser modulo*** $\overset{*}{\vdash\!\dashv}$ *if:*

1. $\rightarrow \cdot \rightsquigarrow^*$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow\ \subseteq\ \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow$
3. *if* $a \vdash\!\dashv b \rightarrow c$ *then* $a \rightarrow \cdot \Rightarrow^* \cdot \overset{=}{\vdash\!\dashv} \cdot {}^*\!\Leftarrow \cdot \leftarrow b$

$$\begin{tikzcd}
 & \cdot \arrow[dl] \arrow[dr] & \\
\cdot \arrow[d, Rightarrow, "*"'] & & \cdot \arrow[d, Rightarrow, "*"] \\
\cdot \arrow[rr, phantom, "\overset{=}{\vdash\dashv}"] & & \cdot
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\cdot \arrow[r, phantom, "\vdash\dashv"] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \\
\cdot \arrow[d, Rightarrow, "*"'] & \cdot \arrow[d, Rightarrow, "*"] & \\
\cdot \arrow[r, phantom, "\overset{=}{\vdash\dashv}"] & \cdot &
\end{tikzcd}$$

Intuition

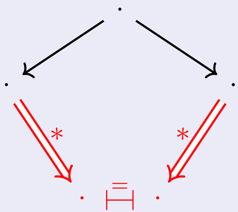
Complexity moved from normalization to diagram closure

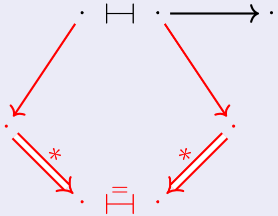
Let's be more subtile

Proposition (inspired from [Aoto and Toyama 2012])

Let $\rightarrow$, $\vdash\!\dashv$ and $\leadsto$ be relations such that $\vdash\!\dashv$ is symmetric and $\leadsto\ \subseteq\ \vdash\!\dashv$.
*Set $\Rightarrow\ =\ \rightarrow\ \cup \leadsto$. Then $\rightarrow$ is **Church-Rosser modulo** $\overset{*}{\vdash\!\dashv}$ if:*

1. $\rightarrow \cdot \leadsto^{*}$ *is strongly normalizing*
2. $\leftarrow \cdot \rightarrow\ \subseteq\ \Rightarrow^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot {}^{*}\!\Leftarrow$
3. *if $a \vdash\!\dashv b \rightarrow c$ then $a \rightarrow \cdot \Rightarrow^{*} \cdot \overset{=}{\vdash\!\dashv} \cdot {}^{*}\!\Leftarrow \cdot \leftarrow b$*





Intuition

Complexity moved from normalization to diagram closure

Step 1 bis: Strong normalization

Proposition

Pose $\xrightarrow{\top}$ = rule commutations without $\top$-commutations in the direction "creating rules" $\cup$ the cut-cut step of cut-elimination

Then $\xrightarrow{\overline{\beta}} \cdot \xrightarrow{\top^*}$ is strongly normalizing, with $\xrightarrow{\overline{\beta}} = (\xrightarrow{\beta}$ without cut-cut$)$

Step 1 bis: Strong normalization

Proposition

Pose $\xrightarrow{\top}$ = *rule commutations without* $\top$*-commutations in the direction "creating rules"* $\cup$ *the cut-cut step of cut-elimination*

Then $\xrightarrow{\overline{\beta}} \cdot \xrightarrow{\top *}$ *is strongly normalizing, with* $\xrightarrow{\overline{\beta}} = (\xrightarrow{\beta}$ *without cut-cut*$)$

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Strong normalization property for second order linear logic

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ABSTRACT

The paper contains the first complete proof of strong normalization (SN) for full second order linear logic (LL): Girard's original proof uses a standardization theorem which is not proven. We introduce sliced pure structures (sps), a very general version of Girard's proof-nets, and we apply to sps Gandy's method to infer SN from weak normalization (WN). We prove a standardization theorem for sps: if WN without erasing steps holds for an sps, then it enjoys SN. A key step in our proof of standardization is a confluence theorem for sps obtained by using only a very weak form of correctness, namely acyclicity slice by slice. We conclude by showing how standardization for sps allows to prove SN of LL, using as usual Girard's reducibility candidates.

1. Introduction

In every abstract approach to computation, the distinction between terminating and non-terminating processes is crucial. A rewriting system enjoys *weak normalization* (WN) if every term of the system can be evaluated in a finite number of steps.

In the λ-calculus, non terminating computations start from λ-terms that strongly exploit *self-application*: every λ-term can be applied to itself (see, for example, [13]). Termination fails for the λ-calculus (even in its weak form WN), but holds for some of its most remarkable subsystems: the simply typed λ-calculus and its extension Girard's system F [6]. The proofs of WN for these calculi have a deep logical content: they correspond to proofs of consistency in the logical sense, as highlighted by the *proofs-as-programs* paradigm. This paradigm is also called *Curry–Howard isomorphism* and establishes a correspondence between a fragment of intuitionistic natural deduction and the typed λ-calculus, so that basically: (i) a type can be seen as a formula (and vice versa); (ii) a λ-term can be seen as a proof (and vice versa); and (iii) the β-reduction of a λ-term can be seen as the application of the cut-elimination procedure to the corresponding proof (and vice versa). The β-redexes correspond to the cuts of the natural deduction (i.e. to the pairs of an introduction rule and an elimination rule of the same connective [19]), hence the β-reduction to a normal form is equivalent to eliminating the cuts in a proof: WN corresponds exactly to cut-elimination. In traditional proof-theory (dating back to Gentzen) cut-elimination is a key property of a logical system, from which the consistency of the system immediately follows[1]: WN of the simply typed λ-calculus (resp. of system F) corresponds then to a consistency proof for the implicational fragment of intuitionistic natural deduction NJ (resp. second order natural deduction NJ[2]).

The proofs-as-programs approach is a way to put constraints on the possibility of building λ-terms, and more precisely on self-application: from the logical point of view, these constraints are sufficient conditions to prove the consistency of

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1 Things went actually the other way round: cut-elimination was proven by Gentzen *in order to* prove consistency of Peano Arithmetic PA.

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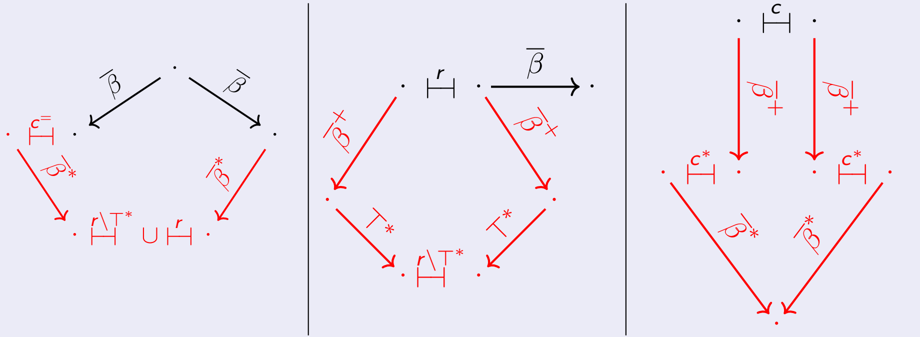
doi:10.1016/j.tcs.2009.07.053

Almost [Pagani and Tortora de Falco 2010]

$\rightsquigarrow$ "Just" check that **some additions for $\top$ at the start** go through this long and technical proof using non-standard proof-nets and non-trivial rewriting results!

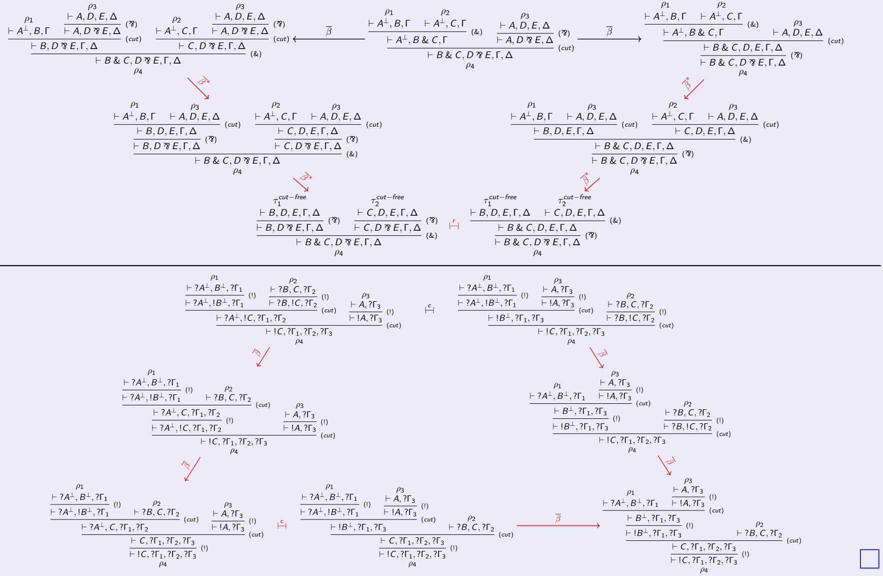
Step 2: Diagrams to close

Lemma



Step 2: Diagrams to close

Proof by cases.



Main Result

Theorem: Church-Rosser modulo *in all fragments of LL^2*

Cut-elimination is ***Church-Rosser modulo*** *rule commutation.*
Hence, if π_1 and π_2 are cut-free with $\pi_1 \xleftarrow{\beta^*} \pi \xrightarrow{\beta^*} \pi_2$ then $\pi_1 \overset{r^*}{\vdash\!\dashv} \pi_2$.

$$\begin{tikzcd} & \pi \arrow[dl, "\beta^*"'] \arrow[dr, "\beta^*"] & \\ \pi_1 \arrow[rr, phantom, "\overset{r^*}{\vdash\!\dashv}"] & & \pi_2 \end{tikzcd}$$

Main Result

Theorem: Church-Rosser modulo *in all fragments of LL^2*

*Cut-elimination is **Church-Rosser modulo** rule commutation.*
Hence, if π_1 and π_2 are cut-free with $\pi_1 \xleftarrow{\beta^*} \pi \xrightarrow{\beta^*} \pi_2$ then $\pi_1 \overset{r^*}{\vdash\!\dashv} \pi_2$.

$$\begin{tikzcd} & \pi \arrow[dl, "\beta^*"'] \arrow[dr, "\beta^*"] & \\ \pi_1 & \overset{r^*}{\vdash\!\dashv} & \pi_2 \end{tikzcd}$$

Theorem: Equality on cut-free proofs *in most fragments of LL^2*

Between cut-free proofs $=_\beta \; = \; \overset{r^*}{\vdash\!\dashv}$

cut-free proofs

Proof.

$\overset{r}{\vdash\!\dashv} \subseteq =_\beta$ is easy (but tedious); if $?c$ needs $\otimes$ $⅋$; if $?w$ needs 1 $\perp$ $\square$

Plan

- Confluence of Cut-elimination Modulo Rule Commutation

$$\begin{tikzcd} & \pi \arrow[dl, "\beta^*"'] \arrow[dr, "\beta^*"] & \\ \pi_1 \arrow[rr, phantom, "\overset{r^*}{\longmapsto\!\!\!\!\dashv}"] & & \pi_2 \end{tikzcd}$$

- Undecidability of Equality up to Cut / Rule Commutation

$$\pi =_\beta \rho \text{ ?}$$

$$\pi \overset{r^*}{\longmapsto\!\!\!\!\dashv} \rho \text{ ?}$$

Undecidability of Cut-Equality/Proof Equivalence

Proof Equivalence: *given π and ρ, does $\pi \overset{r^*}{\longmapsto\!\dashv} \rho$ hold?*

Sub-system	Proof Equivalence	
MLL	PSPACE-complete	[Heijltjes and Houston 2016]
unit-free MALL	LOGSPACE-complete	[Bagnol 2017]

Undecidability of Cut-Equality/Proof Equivalence

Proof Equivalence: *given π and ρ, does $\pi \overset{r^*}{\longmapsto\!\dashv} \rho$ hold?*

Sub-system	Proof Equivalence	
MLL	PSPACE-complete	[Heijltjes and Houston 2016]
unit-free MALL	LOGSPACE-complete	[Bagnol 2017]
LL	Undecidable	[NEW]
LL^2 without $\top$	Decidable	(finite number of proofs in a class)
MALL	Decidable	(finite number of cut-free proofs)

Key Lemma *Proof next slide*

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r^*}{\longmapsto\!\dashv} \quad \dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \text{ is provable}$$

Proposition

Proof Equivalence (so equality up to cut) is undecidable in propositional LL

Proof.

Were it decidable, provability would be too but it is not [Lincoln 1995] $\square$

Provability reduces to Proof Equivalence

Key Lemma

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_1) \quad \overset{r^*}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash\, !A \otimes \top, \top}\,(\top)}{\vdash\, !A \otimes \top, \top \oplus \top}\,(\oplus_2) \iff A \text{ is provable}$$

Proof.

♦ If A is provable

We use its proof to find a sequence of commutations:

$$\dfrac{\dfrac{}{\vdash\, !A \otimes \top_A, \top}\,(\top)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\dfrac{\vdash A}{\vdash\, !A}\,(!) \qquad \dfrac{}{\vdash \top_A, \top}\,(\top)}{\vdash\, !A \otimes \top_A, \top}\,(\otimes)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\dfrac{\vdash A}{\vdash\, !A}\,(!) \qquad \dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash\, !A \otimes \top_A, \top}\,(\otimes)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\oplus_i)$$

$$\overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\vdash A}{\vdash\, !A}\,(!) \qquad \dfrac{\dfrac{}{\vdash \top_A, \top}\,(\top_A)}{\vdash \top_A, \top \oplus \top}\,(\oplus_i)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\otimes) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\vdash A}{\vdash\, !A}\,(!) \qquad \dfrac{}{\vdash \top_A, \top \oplus \top}\,(\top_A)}{\vdash\, !A \otimes \top_A, \top \oplus \top}\,(\otimes)$$

Provability reduces to Proof Equivalence

Key Lemma

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\ (\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\ (\oplus_1) \quad \overset{r^*}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{}{\vdash\ !A \otimes \top, \top}\ (\top)}{\vdash\ !A \otimes \top, \top \oplus \top}\ (\oplus_2) \iff A \textit{ is provable}$$

Proof.

♦ **If A is provable**

♦ **If A is not provable**

We can compute the full equivalence class in this case:

$$\dfrac{\dfrac{}{\vdash\ !A \otimes \top_A, \top}\ (\top)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\ (\oplus_i) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\ (\top) \qquad \dfrac{}{\vdash \top_A}\ (\top_A)}{\vdash\ !A \otimes \top_A, \top}\ (\otimes)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\ (\oplus_i) \quad \overset{r}{\longmapsto\!\!\dashv} \quad \dfrac{\dfrac{\dfrac{}{\vdash\ !A, \top}\ (\top)}{\vdash\ !A, \top \oplus \top}\ (\oplus_i) \qquad \dfrac{}{\vdash \top_A}\ (\top_A)}{\vdash\ !A \otimes \top_A, \top \oplus \top}\ (\otimes)$$

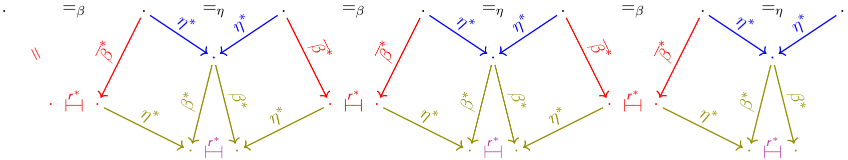
Intuition

Use $!A$ to have no commutation in $\dfrac{}{\vdash\ !A, \top}\ (\top)$ (! commutes not with $\top$)

□

One may want for more!

- **Axiom-expansion** and its *interactions* with cut-elimination



One may want for more!

- **Axiom-expansion** and its *interactions* with cut-elimination
- **More commutations** with *new cases* to check

$$\dfrac{\dfrac{\vdash A, ?B, ?B, ?\Gamma}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \quad\equiv\quad \dfrac{\dfrac{\vdash A, ?B, ?B, ?\Gamma}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \quad\equiv\quad \dfrac{\dfrac{\vdash A, ?\Gamma}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$

$$\dfrac{\begin{array}{c}\pi_B\\ \vdash A[B/X], \Gamma\end{array}}{\vdash \exists X\ A, \Gamma}\,(\exists) \quad\equiv\quad \dfrac{\begin{array}{c}\pi_C\\ \vdash A[C/X], \Gamma\end{array}}{\vdash \exists X\ A, \Gamma}\,(\exists)$$

when π_B and π_C are "witness irrelevant"

One may want for more!

- **Axiom-expansion** and its *interactions* with cut-elimination
- **More commutations** with *new cases* to check
- **More rewriting rules** with *interactions* to check

$$\dfrac{\dfrac{\vdash ?A, \Gamma}{\vdash ?A, ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(?_c) \quad \longrightarrow \quad \vdash ?A, \Gamma$$

One may want for more!

- **Axiom-expansion** and its *interactions* with cut-elimination
- **More commutations** with *new cases* to check
- **More rewriting rules** with *interactions* to check

$$\dfrac{\dfrac{\vdash ?A, \Gamma}{\vdash ?A, ?A, \Gamma}\ (?_w)}{\vdash ?A, \Gamma}\ (?_c) \quad \longrightarrow \quad \vdash ?A, \Gamma$$

Thank you!

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Back-up: Isomorphisms in Linear Logic

Isomorphism $A \simeq B$

Proofs π of $A \vdash B$ and ρ of $B \vdash A$ such that

$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A ⅋ (B ⅋ C) \simeq (A ⅋ B) ⅋ C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A ⅋ B \simeq B ⅋ A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A ⅋ \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A ⅋ (B \& C) \simeq (A ⅋ B) \& (A ⅋ C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A ⅋ \top \simeq \top$	
Seely	$!(A \& B) \simeq !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq ?A ⅋ ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \& B) \simeq \forall X\ A \& \forall X\ B$ $\forall X\ B ⅋ A \simeq \forall X(B ⅋ A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$

$*$ if X not free in A

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$$\dfrac{\overset{\pi}{A \vdash B} \qquad \overset{\rho}{B \vdash A}}{A \vdash A}\,(cut) =_{\beta\eta o} \dfrac{}{A \vdash A}\,(ax) \quad \text{and} \quad \dfrac{\overset{\rho}{B \vdash A} \qquad \overset{\pi}{A \vdash B}}{B \vdash B}\,(cut) =_{\beta\eta o} \dfrac{}{B \vdash B}\,(ax)$$

Conjecture

Associativity	$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$ $A \oplus (B \oplus C) \simeq (A \oplus B) \oplus C$		$A \parr (B \parr C) \simeq (A \parr B) \parr C$ $A \& (B \& C) \simeq (A \& B) \& C$	
Commutativity	$A \otimes B \simeq B \otimes A$	$A \parr B \simeq B \parr A$	$A \oplus B \simeq B \oplus A$	$A \& B \simeq B \& A$
Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \& \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) \simeq (A \parr B) \& (A \parr C)$	
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$*$ if X not free in A

With the $! - ?_c$, $! - ?_w$ and $?_c - ?_w$ commutations and reductions

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \equiv \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{\vdash ?A, \Gamma}}{\vdash ?A, ?A, \Gamma}\,(?_w)}{\vdash ?A, \Gamma}\,(?_c) \to \overset{\pi}{\vdash ?A, \Gamma}$$

Back-up: Isomorphisms in Linear Logic

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Neutrality	$A \otimes 1 \simeq A$	$A \parr \bot \simeq A$	$A \oplus 0 \simeq A$	$A \,\&\, \top \simeq A$
Distributivity	$A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \,\&\, C) \simeq (A \parr B) \,\&\, (A \parr C)$	
Annihilation	$A \otimes 0 \simeq 0$		$A \parr \top \simeq \top$	
Seely	$!(A \,\&\, B) \simeq\, !A \otimes !B$ $!\top \simeq 1$		$?(A \oplus B) \simeq\, ?A \parr ?B$ $?0 \simeq \bot$	
Quantifiers	$\forall X(A \,\&\, B) \simeq \forall X\ A \,\&\, \forall X\ B$ $\forall X\ B \parr A \simeq \forall X(B \parr A)^*$	$\exists X(A \oplus B) \simeq \exists X\ A \oplus \exists X\ B$ $\exists X\ B \otimes A \simeq \exists X(B \otimes A)^*$	$\forall X\ \top \simeq \top$ $\forall X \forall Y\ A \simeq \forall Y \forall X\ A$	$\exists X\ 0 \simeq 0$ $\exists X \exists Y\ A \simeq \exists Y \exists X\ A$
Optional	$\forall X\ A \simeq A^{*\dagger}$	$\exists X\ A \simeq A^{*\dagger}$	$1 \simeq \bot^{\ddagger}$	$0 \simeq \top^{\clubsuit}$

$*$ if X not free in A

$\dagger$ if $\dfrac{\overset{\pi_B}{\vdash A[B/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists) \equiv \dfrac{\overset{\pi_C}{\vdash A[C/X], \Gamma}}{\vdash \exists X\ A, \Gamma}\,(\exists)$

when π is "witness irrelevant"

$\ddagger$ if $\dfrac{\overset{\pi}{\vdash \Gamma} \qquad \dfrac{}{\vdash}\,(mix_0)}{\vdash \Gamma}\,(mix_2) \equiv \overset{\pi}{\vdash \Gamma}$

$\clubsuit$ with $\dfrac{}{\vdash \Gamma}\,(\emptyset)$

Back-up: Inference Rules of Linear Logic

$$\frac{}{\vdash A^\perp, A}\ (ax) \qquad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma}\ (\perp) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \mathbin{\&} B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

$$X \text{ not free in } \Gamma \quad \frac{\vdash A, \Gamma}{\vdash \forall X\ A, \Gamma}\ (\forall) \qquad \frac{\vdash A\{B/X\}, \Gamma}{\vdash \exists X\ A, \Gamma}\ (\exists)$$

Back-up: Cut-elimination on an example

$$
\dfrac{
\dfrac{
\dfrac{}{\vdash A^\perp, A}\,(ax)
\qquad
\dfrac{}{\vdash B, B^\perp}\,(ax)
}{\vdash A^\perp, A \otimes B, B^\perp}\,(\otimes)
\qquad
\dfrac{
\dfrac{}{\vdash A, A^\perp}\,(ax)
\qquad
\dfrac{}{\vdash C^\perp, C}\,(ax)
}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)
}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(cut)
$$

$$\xrightarrow{\beta_{com}}$$

$$
\dfrac{
\dfrac{
\dfrac{}{\vdash A^\perp, A}\,(ax)
\qquad
\dfrac{
\dfrac{}{\vdash A, A^\perp}\,(ax)
\qquad
\dfrac{}{\vdash C^\perp, C}\,(ax)
}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)
}{\vdash A, A^\perp \otimes C^\perp, C}\,(cut)
\qquad
\dfrac{}{\vdash B, B^\perp}\,(ax)
}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(\otimes)
$$

$$\xrightarrow{\beta_{key}}$$

$$
\dfrac{
\dfrac{
\dfrac{}{\vdash A, A^\perp}\,(ax)
\qquad
\dfrac{}{\vdash C^\perp, C}\,(ax)
}{\vdash A, A^\perp \otimes C^\perp, C}\,(\otimes)
\qquad
\dfrac{}{\vdash B, B^\perp}\,(ax)
}{\vdash A \otimes B, A^\perp \otimes C^\perp, B^\perp, C}\,(\otimes)
$$

Back-up: Rule commutation (list of cases)

$$\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \dfrac{\overset{\rho}{\vdash A, D, \Delta} \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, D, \Delta, \Sigma}\,(\otimes)}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\overset{\pi}{\vdash C, \Gamma} \qquad \overset{\rho}{\vdash A, D, \Delta}}{\vdash A, C \otimes D, \Gamma, \Delta}\,(\otimes) \qquad \overset{\tau}{\vdash B, \Sigma}}{\vdash A \otimes B, C \otimes D, \Gamma, \Delta, \Sigma}\,(\otimes)$$

$$\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \dfrac{\overset{\rho}{\vdash B, C, \Delta} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash B, C \,\&\, D, \Delta}\,(\&)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\otimes) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\rho}{\vdash B, C, \Delta}}{\vdash A \otimes B, C, \Gamma, \Delta}\,(\otimes) \qquad \dfrac{\overset{\pi}{\vdash A, \Gamma} \qquad \overset{\tau}{\vdash B, D, \Delta}}{\vdash A \otimes B, D, \Gamma, \Delta}\,(\otimes)}{\vdash A \otimes B, C \,\&\, D, \Gamma, \Delta}\,(\&)$$

$$\dfrac{}{\vdash \top, ?A, \Gamma}\,(\top) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash \top, ?A, ?A, \Gamma}\,(\top)}{\vdash \top, ?A, \Gamma}\,(?_c)$$

$$\dfrac{}{\vdash \top, A \otimes B, \Gamma, \Delta}\,(\top) \quad \overset{r}{\vdash\!\dashv} \quad \dfrac{\dfrac{}{\vdash \top, A, \Gamma}\,(\top) \qquad \overset{\pi}{\vdash B, \Delta}}{\vdash \top, A \otimes B, \Gamma, \Delta}\,(\otimes)$$

... and many many many more ...

- ! Non-trivial: duplicates / merges sub-proofs
- !! Tricky: produces / deletes rules and sub-proofs

Back-up: Rule commutation (general method)

Every pair $\dfrac{\dfrac{}{\quad}\,(s)}{\quad}\,(r) \;\overset{r}{\vdash\!\dashv}\; \dfrac{\dfrac{}{\quad}\,(r)}{\quad}\,(s)$ coming from:

$$\dfrac{\dfrac{\dfrac{}{\quad}\,(s)}{\quad}\,(r)}{\qquad\qquad}\,(cut) \xleftarrow{\beta^*_{com}} \dfrac{\dfrac{}{\quad}\,(r) \qquad \dfrac{}{\quad}\,(s)}{\qquad\qquad}\,(cut) \xrightarrow{\beta^*_{com}} \dfrac{\dfrac{\dfrac{}{\quad}\,(r)}{\quad}\,(s)}{\qquad\qquad}\,(cut)$$

$\approx \#|rules|^2$ commutations $\longrightarrow$ 93 equations in LL!

Remarks

- $\overset{r}{\vdash\!\dashv} \subseteq =_\beta$
- $\overset{r}{\vdash\!\dashv}$ is the usual (cut-free) commutation without $! - ?_c$ and $! - ?_w$

$$\dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash !A, ?B, ?B, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_c) \;\;\overset{r}{\not\vdash\!\dashv}\;\; \dfrac{\dfrac{\overset{\pi}{\vdash A, ?B, ?B, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_c)}{\vdash !A, ?B, ?\Gamma}\,(!) \quad \text{and} \quad \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash !A, ?\Gamma}\,(!)}{\vdash !A, ?B, ?\Gamma}\,(?_w) \;\;\overset{r}{\not\vdash\!\dashv}\;\; \dfrac{\dfrac{\overset{\pi}{\vdash A, ?\Gamma}}{\vdash A, ?B, ?\Gamma}\,(?_w)}{\vdash !A, ?B, ?\Gamma}\,(!)$$