A Formalization of Multiplicative Proof-Nets in Rocq

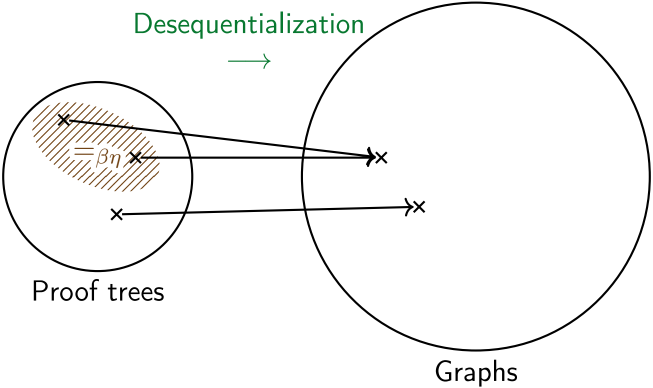
Rémi Di Guardia* Olivier Laurent†

*Paris, †Lyon

TLLA 2025, 19 July 2025

Introduction

Proof nets: graphical syntax for proofs of **Linear Logic**, *canonical*



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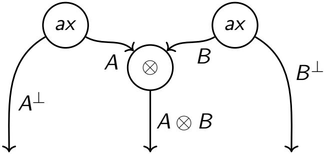

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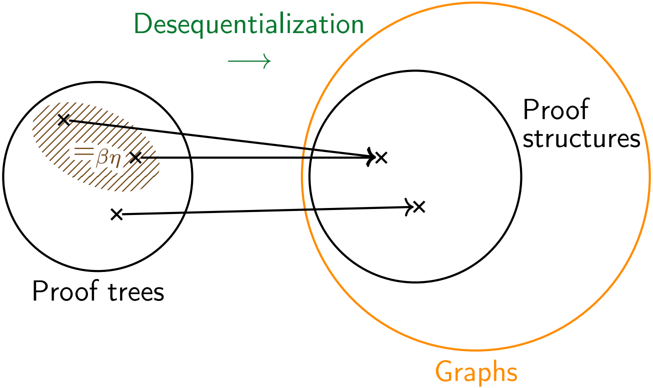

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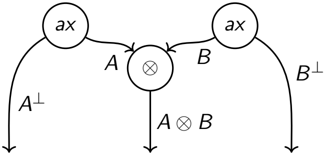

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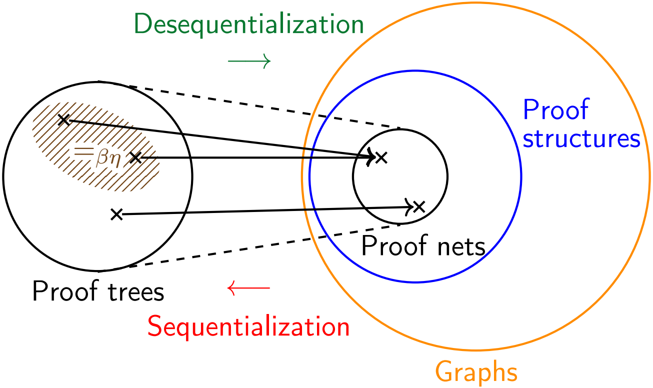

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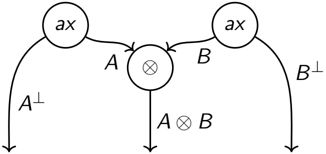

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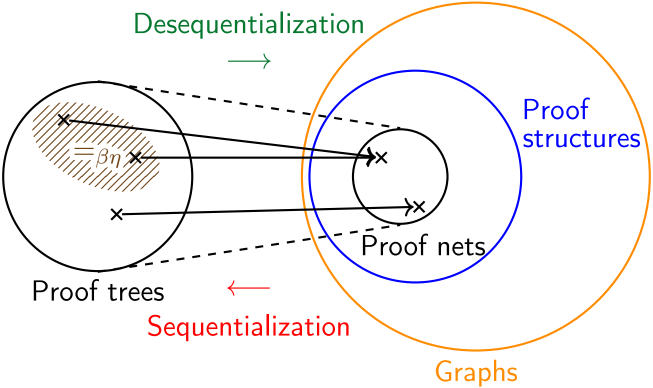

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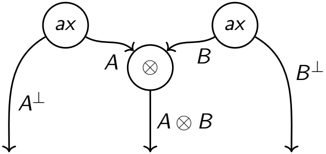


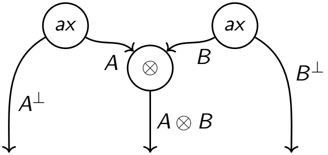
This talk: how to formalize proof nets in a proof assistant?

Around Formalizations of Linear Logic

Already many formalizations of linear logic in ...

- Rocq [Lau17; PW99; Xav+18; Bos+11; Péd; Sad03]
- Abella [CLR19; CLR17]
- Isabelle [KP95; Gro95]

... but always for *sequent calculus* and **never** for *proof nets*!



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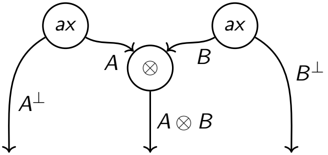
What is the problem?

- Manipulations of **multi/hyper-graphs** and their isomorphisms, **non inductive** syntax
- **Complex** + **Several definitions** with strata – structures and nets



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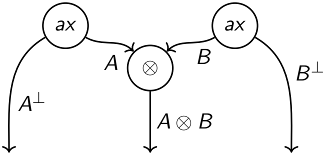
- **Complex** + **Several definitions** with strata – structures and nets
- **Geometric/Graphical/Implicit** arguments, with little drawings



Motivations & Goals

Why formalize proof nets now?

- **Because** of the liberties taken on paper!
- First step towards **complicated** proof nets (additive, first order, . . .)
- GraphTheory library in ROCQ (1st release in **2020**)
  - proves difficult results – treewidth, minors, . . .
  - **multigraphs** with needed operations: adding/removing vertices and edges, sub-graphs, isomorphisms, . . .

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What did we formalize?

- **definition** of proof nets
- **desequentialization** from sequents to graphs + it yields a proof net
- **sequentialization**: proof-nets $\simeq$ images of desequentialization
- **cut-elimination** + gives proof net

Outline

- **Unit-free Multiplicative Linear Logic**
- **Formalization of Proof Nets**
  - Underlying Graphs
  - Correctness Criterion

Unit-free Multiplicative Linear Logic

Formulas

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A$$

Orthogonality

$$(X^{\perp})^{\perp} = X \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

Sequents (lists)

$$\vdash A_1, A_2, \ldots, A_n$$

Rules

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A^{\perp}, \Gamma \qquad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\ (ex)$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr)$$

Desequentialization

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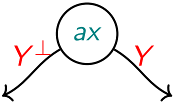

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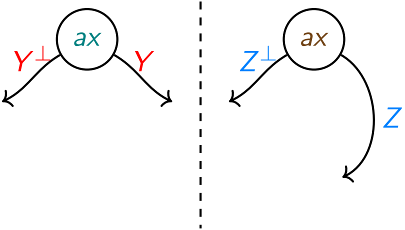

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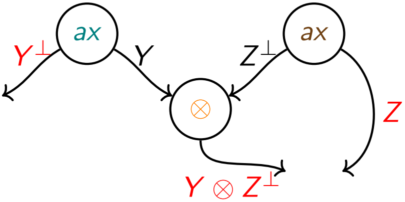

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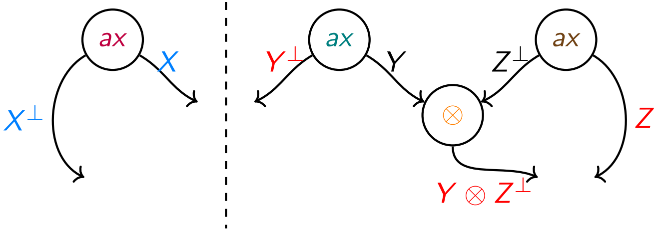

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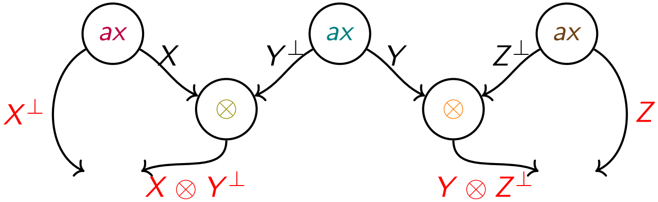

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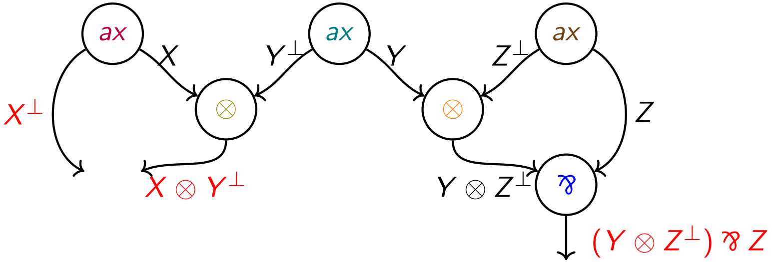

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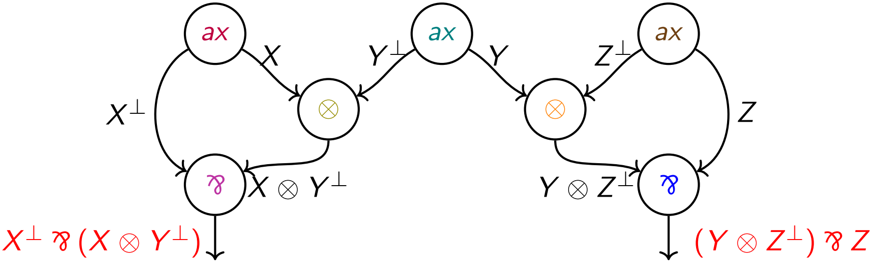
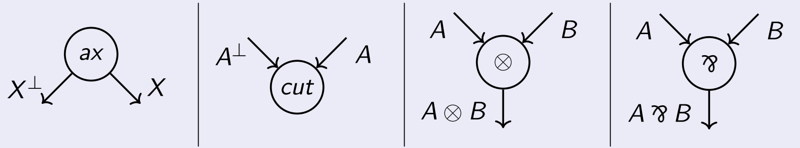


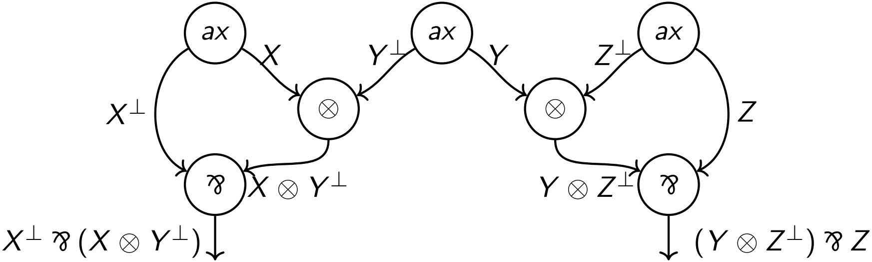
Proof Structure

Definition

Partial directed multigraph with labels on vertices $\rightarrow$ *ax* / *cut* / $\otimes$ / $\parr$

on edges $\rightarrow$ formula





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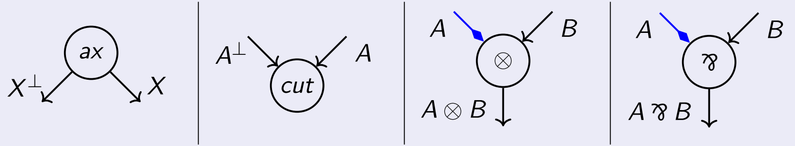

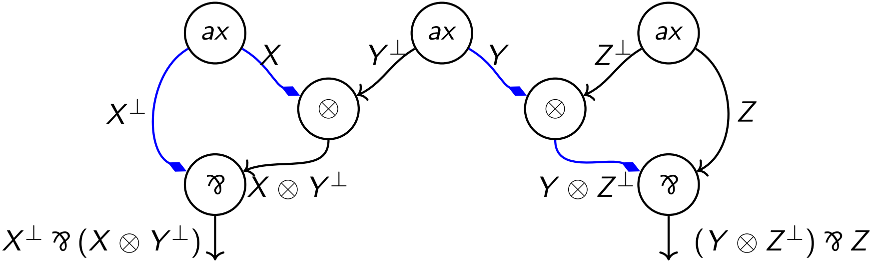

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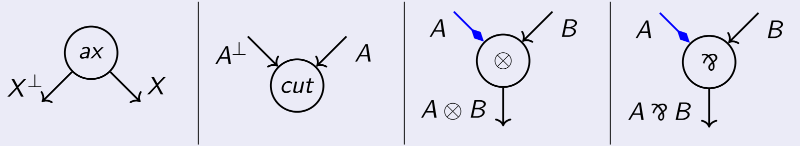

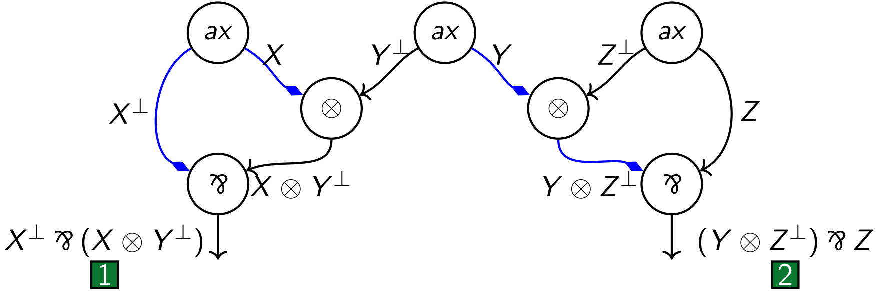

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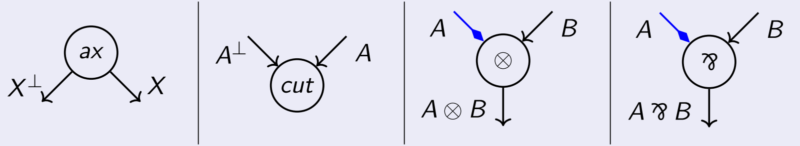

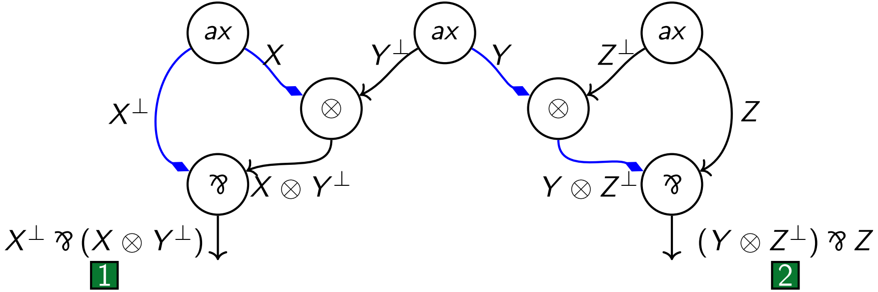

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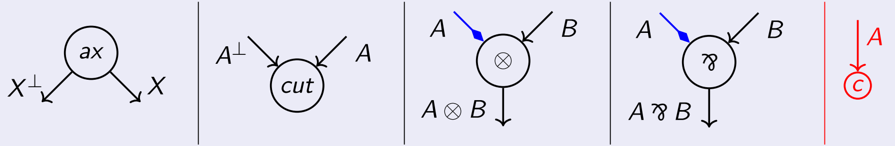
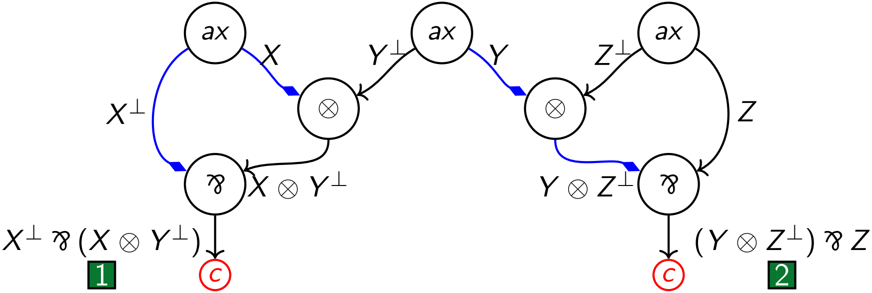




Correctness Criterion & Proof Net

Definition: Correctness Graph

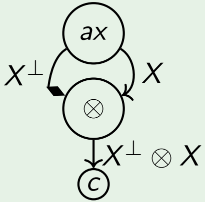
Remove one in-edge of each $⅋$, forget the orientation of edges

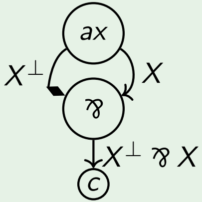
Definition: Correctness Criterion (Danos-Regnier)

Correct $=$ all correctness graphs are **acyclic** and **connected**

Proof Nets $=$ correct proof structures

Toy Examples





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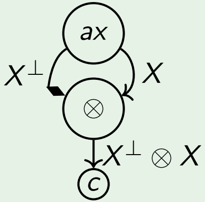
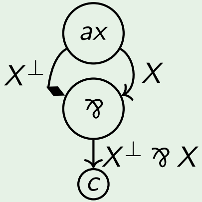


not acyclic (but connected)

INCORRECT



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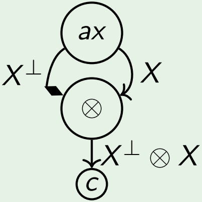

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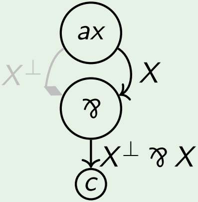

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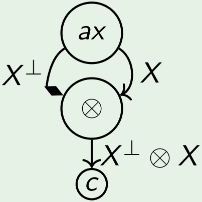

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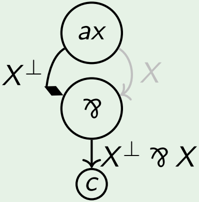

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CORRECT

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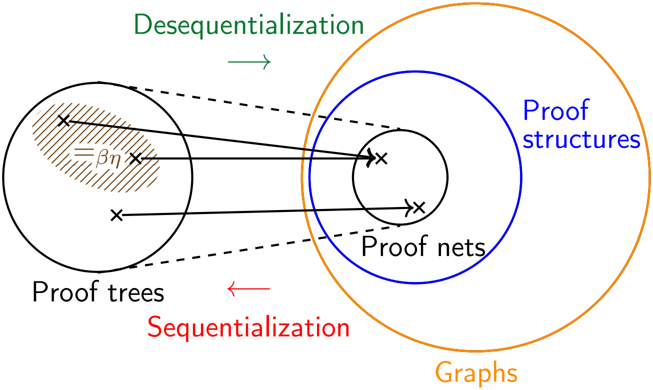
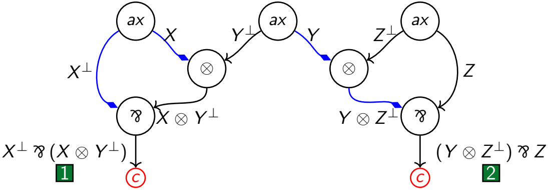
- **Formalization of Proof Nets**
  - Underlying Graphs
  - Correctness Criterion

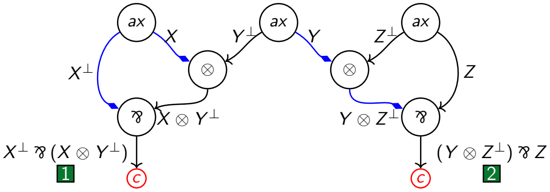




Direct Traduction:

```
Record graph_data : Type :=
  Graph_data {
    graph_of :> graph rule formula;
     left : { v : vertex graph_of | vlabel v == ⊗ || vlabel v == ⅋ }
          → edge graph_of;
     right : { v : vertex graph_of | vlabel v == ⊗ || vlabel v == ⅋ }
          → edge graph_of;
     order : { v : vertex graph_of | vlabel v == c } →
          'I_#|{ v : vertex graph_of | vlabel v == c }|;
  }.
```

Implementation of the underlying graphs



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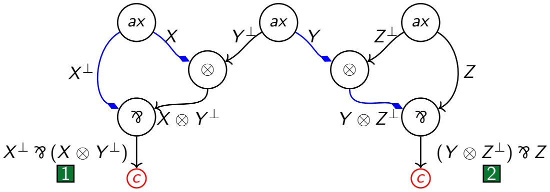
```

$\longrightarrow$ Dependant types quickly too complex

$\rightsquigarrow$ To **define** adding a vertex, need to **prove** that each $\otimes$ before is still a $\otimes$ after

Implementation of the wanted graphs bis

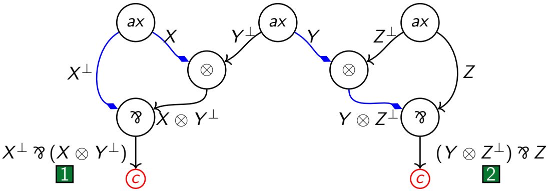


Adopted Solution:

Record *graph_data* : Type :=
  Graph_data {
    graph_of :> *graph rule formula*;
    left : *edge graph_of* → *bool*;
    order : *seq* (*vertex graph_of*);
  }.

- A boolean to mark left arrows
- A list of vertices for ordering the conclusions
- Properties to check:
  - no two marked edges for a same vertex
  - the list contains exactly c-vertices
  - ...

Implementation of the wanted graphs bis



Adopted Solution:

```
Record graph_data : Type :=
  Graph_data {
    graph_of :> graph rule formula;
    left : edge graph_of → bool;
    order : seq (vertex graph_of);
  }.
```

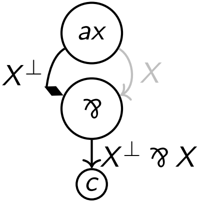
- A boolean to mark left arrows
- A list of vertices for ordering the conclusions
- Properties to check:
  - no two marked edges for a same vertex
  - the list contains exactly c-vertices
  - ...

$\longrightarrow$ Remove difficulties from **definitions** to put them in the **proofs**

Formalization of the Correctness Criterion

Reminder

Correctness Graph: remove one in-edge of each $\parr$, forget the orientation
Correctness: all correctness graphs are **acyclic** and **connected**



On computer: **horrible!**

$\longrightarrow$ correctness graphs of $G + v \simeq$ (correctness graphs of G) $+ v$

$\longrightarrow$ not the same edges in G and in its correctness graphs

$\Longrightarrow$ Not trivial to show that adding v preserves correctness!

Correctness Criterion without Correctness Graphs

Idea: correctness directly in the **proof structure**

Acyclicity

$\iff$ every (undirected) cycle uses both in-edges of a same $\parr$

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Lemma

$acyclic \implies \#cc = \#vertices - \#edges$

$\longrightarrow$ all correctness graphs connected iff the one without left edges is

Connectedness

$\iff$ every pair of vertices are linked by a path not using left in-edges of $\parr$

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Connectedness

$\iff$ every pair of vertices are linked by a path not using left in-edges of $\parr$

$\rightsquigarrow$ Study of *particular paths*:

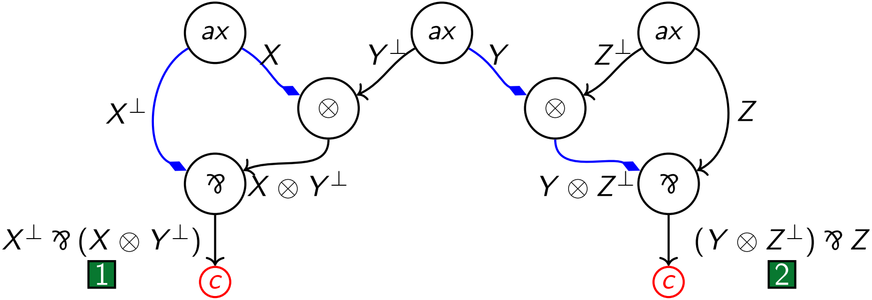
- some edges are incompatible
- some edges are forbidden

Conclusion & Perspectives

What do we have now? *What do we want?*

- **definition** of proof nets
- **desequentialization** from sequents to graphs + it yields a proof net
- **sequentialization**: proof-nets $\simeq$ images of desequentialization
  $\longrightarrow$ the one presented at FSCD yesterday [Di+25]
- **cut-elimination** + gives proof net + convergence + same as sequents
- idem for **axiom-expansion**
- quotient by **rule commutations**
- theory of proof nets: kingdoms, empires
- theorems using proof nets, *e.g.* isomorphisms of MLL [BD99]
- Proof nets for larger systems: MELL, MALL
- ...
- Intermediate results not in **GraphTheory** (undirected paths, etc)

Thank you!



ax
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X
$Y^\perp$
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$\otimes$
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Z
$⅋$
$X \otimes Y^\perp$
$Y \otimes Z^\perp$
$⅋$
$X^\perp ⅋ (X \otimes Y^\perp)$
$(Y \otimes Z^\perp) ⅋ Z$
1
c
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2

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[2] https://github.com/ComputerAidedLL/PowerWebster_ILL

[3] https://github.com/meta-logic/coq-ll

[4] https://github.com/Matafou/ill_narratives

[5] https://github.com/ppedrot/ll-coq

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[7] https://github.com/meta-logic/abella-reasoning

[8] https://www.cl.cam.ac.uk/~lp15/papers/Workshop/papers/kalvala-linear.pdf

[9] https://link.springer.com/chapter/10.1007/3-540-59338-1_41

Strata of definitions for proof nets in Rocq (1/2)

```
Notation base_graph := (graph (flat rule) (flat (formula × bool))).
Definition flabel {G : base_graph} (e : edge G) : formula :=
  fst (elabel e).
Definition llabel {G : base_graph} (e : edge G) : bool :=
  snd (elabel e).

Record graph_data : Type :=
  Graph_data {
    graph_of :> base_graph;
    order : seq (edge graph_of);
  }.

Definition sequent (G : graph_data) : seq formula :=
  [seq flabel e | e ← order G].
```

Strata of definitions for proof nets in Rocq (2/2)

```
Record proof_structure : Type := Proof_structure {
    graph_data_of :> graph_data;
    p_deg : proper_degree graph_data_of;
    p_ax_cut : proper_ax_cut graph_data_of;
    p_tens_parr : proper_tens_parr graph_data_of;
    p_noleft : proper_noleft graph_data_of;
    p_order_full : proper_order_full graph_data_of;
    p_order_uniq : proper_order_uniq graph_data_of;
  }.
```

```
Definition proper_tens_parr (G : base_graph) :=
  ∀ (b : bool) (v : G), vlabel v = (if b then ⅋ else ⊗) →
  ∃ el er ec, el \in edges_at_in v ∧ llabel el ∧
    er \in edges_at_in v ∧ ¬llabel er ∧ ec \in edges_at_out v ∧
    flabel ec = (if b then ⅋ else ⊗) (flabel el) (flabel er).
```

```
Record proof_net : Type := ...
```

f-simple Paths

Study undirected paths respecting some conditions:

- **Acyclicity** $\longrightarrow$ some edges are incompatible
- **Connectedness** $\longrightarrow$ some edges are forbidden

$\implies$ new notion of undirected paths

f-simple Paths

Study undirected paths respecting some conditions:

- **Acyclicity** $\longrightarrow$ some edges are incompatible
- **Connectedness** $\longrightarrow$ some edges are forbidden

$\Longrightarrow$ new notion of undirected paths

f-simple Paths

Given an edge-coloring $f : E \longrightarrow I \cup \{\perp\}$, a path p is f-simple when:

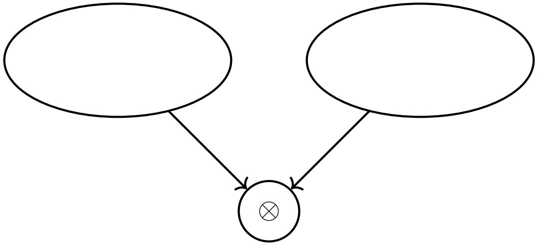
- f is injective on the edges of p
- no edge of p has the forbidden color $\perp$ has an image for f

Difficulties

- Relatively young graph library, lacks some **concepts** for multigraphs: undirected paths, . . .
- **Explicit** manipulations of graphs and their isomorphisms
- Formalizing graph theory reasonnings and their **implicit** arguments

Explicit manipulations of graphs

Proof of Sequentialization: find a splitting vertex almost as easily as on paper, but concluding from there is way more complex!



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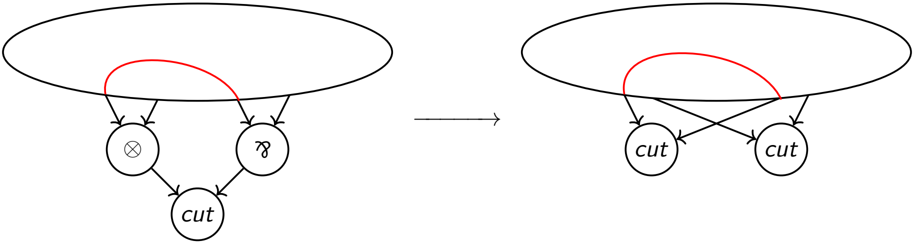
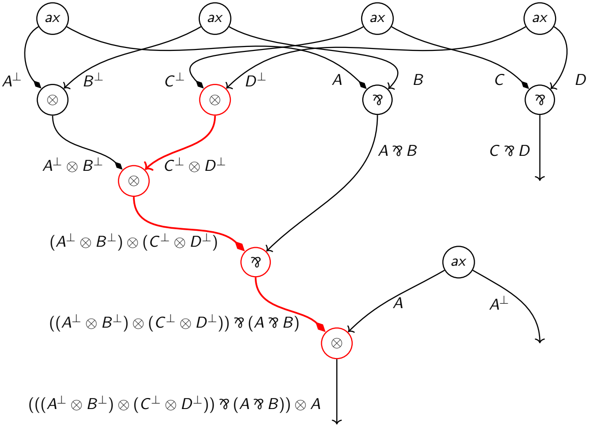


Transfert of paths when adding or removing a vertex, ...

Informal reasoning in graph theory

Descending path

A *descending path* is one whose target has only out-edges towards c-vertices



Formal reasonning in graph theory

Lemma

A proof structure is a DAG (directed acyclic multigraph)

Lemma

The relation "being linked by a directed path" is well-founded in a DAG

Proposition

For a vertex (not c) in a proof structure, there exists a directed path from it to a vertex whose out-edges are all towards c-vertices

Implementation of the wanted graphs bis

Without Dependant Type:

```
Record graph_data : Type :=
  Graph_data {
    graph_of :> graph rule formula;
    left : vertex graph_of → edge graph_of;
    right : vertex graph_of → edge graph_of;
    order : vertex graph_of → int;
  }.
```

Implementation of the wanted graphs bis

Without Dependant Type:

```
Record graph_data : Type :=
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```

⟶ Giving arbitrary values for irrelevant arguments can be the longest part of some definitions / proofs!

Implementation of the wanted graphs bis

Without Dependant Type:

```
Record graph_data : Type :=
  Graph_data {
    graph_of :> graph rule formula;
    left : vertex graph_of → option edge graph_of;
    right : vertex graph_of → option edge graph_of;
    order : vertex graph_of → option int;
  }.
```

⟶ Giving arbitrary values for irrelevant arguments can be the longest part of some definitions / proofs!

⟶ With option types get a boring pattern matching everywhere