Yeo's Theorem for Locally Colored Graphs: the Path to Sequentialization in Linear Logic

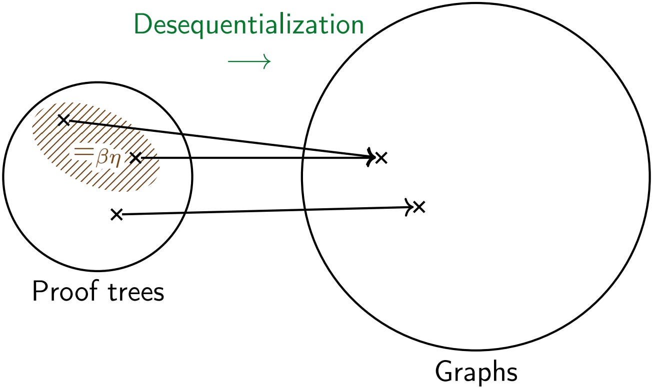
Rémi Di Guardia* Olivier Laurent†
Lorenzo Tortora de Falco‡ Lionel Vaux Auclair§

*Paris, †Lyon, ‡Rome, §Marseille

FSCD 2025, 18 July 2025

Introduction

Proof nets: graphical syntax for proofs of **Linear Logic**, *canonical*



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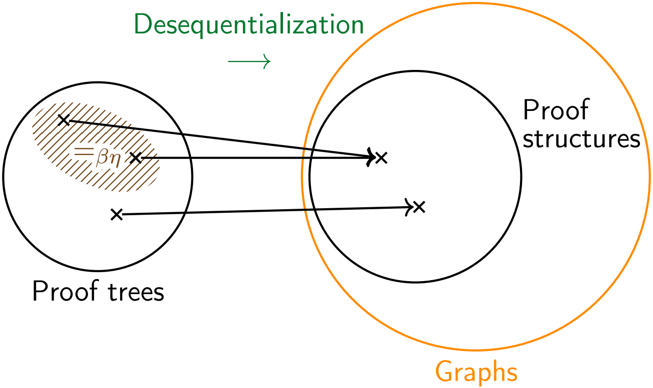

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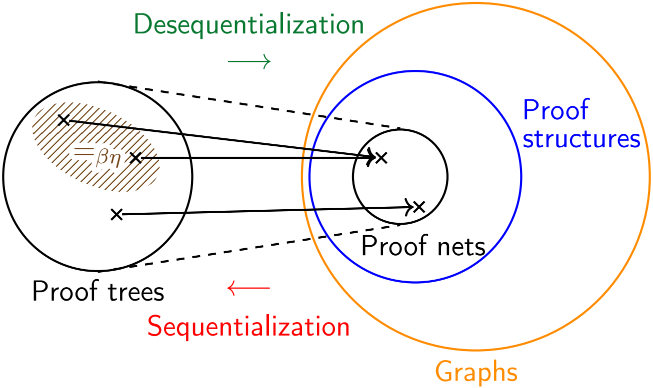

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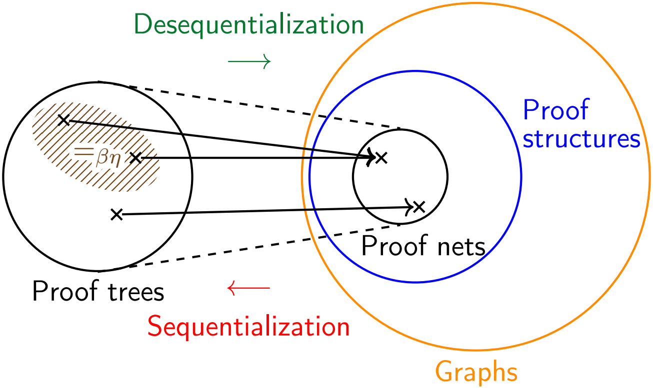

Multiple **correctness criteria**, proofs of sequentialization

Still sequentialization is not considered easy.

Introduction

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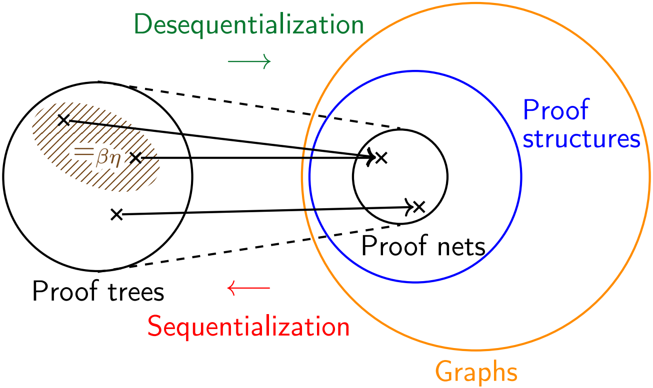

This talk: easy proof of sequentialization using (a generalization of) Yeo's theorem from **graph theory**

$\longrightarrow$ follows a line of work from [Rétoré2003] and [Nguyễn2020]

Outline

- **Multiplicative Linear Logic & Sequentialization**
  - Sequent Calculus & Proof Nets
  - Sequentialization from Yeo's theorem

- **Simple proof of (a generalized) Yeo's theorem**

Unit-free Multiplicative Linear Logic with Mix

Formulas

$$A ::= \underbrace{X \mid \neg X}_{atoms} \mid A \;\wedge\; A \mid A \;\vee\; A$$

Rules

$$\frac{}{\vdash \neg X, X}\ (ax) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \;\wedge\; B, \Gamma, \Delta}\ (\wedge) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \;\vee\; B, \Gamma}\ (\vee)$$

$$\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}\ (mix_2) \qquad \frac{}{\vdash}\ (mix_0)$$

Unit-free Multiplicative Linear Logic with Mix

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Unit-free Multiplicative Linear Logic with Mix

Formulas

$$A ::= \underbrace{X \mid \overset{not}{X^\perp}}_{atoms} \mid A \overset{and}{\otimes} A \mid A \overset{or}{\parr} A$$

Rules

$$\frac{}{\vdash X^\perp, X}\ (ax) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr)$$

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No contraction nor weakening, consistent logic

Example of proof structure by desequentialization

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X}\ (ax) \quad \dfrac{}{\vdash Y, Y^\perp}\ (ax)}{\vdash X \otimes Y, X^\perp, Y^\perp}\ (\otimes) \quad \dfrac{}{\vdash Z, Z^\perp}\ (ax)}{\vdash X \otimes Y, X^\perp, Y^\perp, Z, Z^\perp}\ (mix_2)}{\vdash X \otimes Y, X^\perp \parr Y^\perp, Z, Z^\perp}\ (\parr)}{\vdash X \otimes Y, (X^\perp \parr Y^\perp) \parr Z, Z^\perp}\ (\parr)$$

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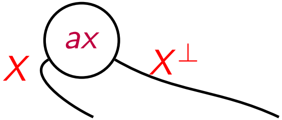

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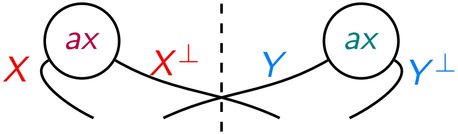

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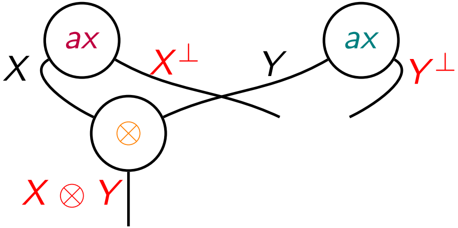

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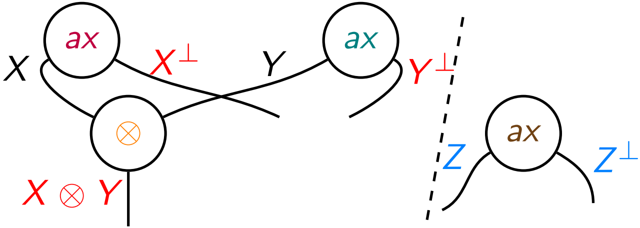

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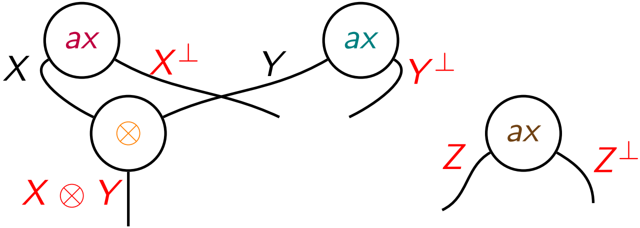

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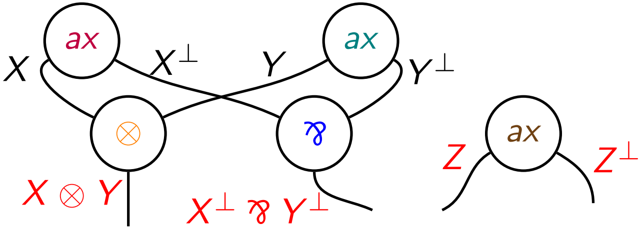

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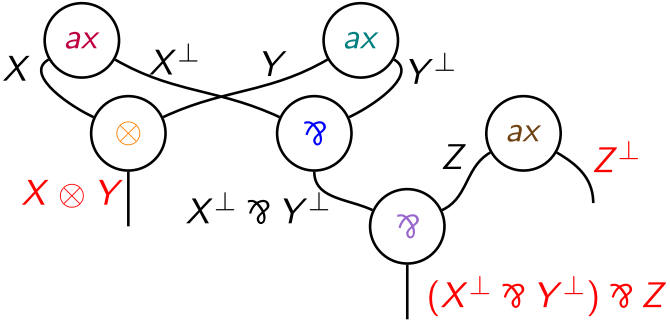
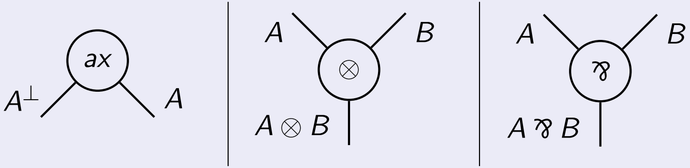
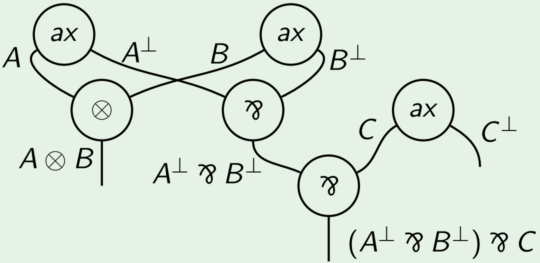


Proof structure

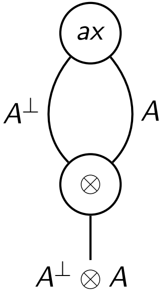
Definition

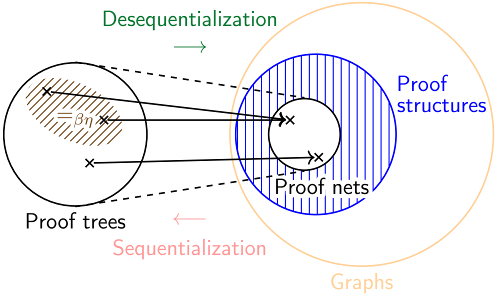
Partial multigraph with labels on vertices → *ax* / $\otimes$ / $\parr$
on edges → formula





Correctness





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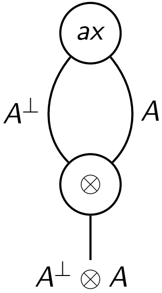
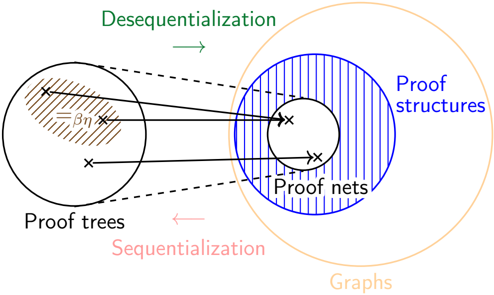
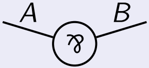




Danos-Regnier Correctness Criterion

Cusp: a ⅋ and its two premises

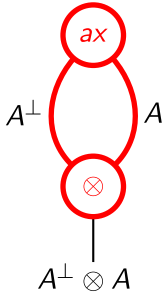


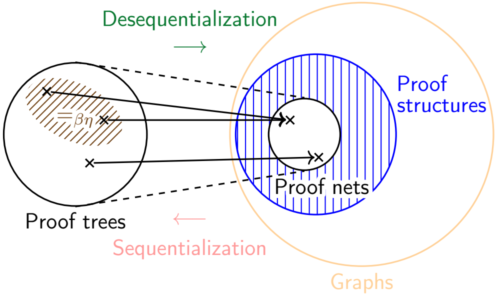
Switching / Cusp-free cycle: does not contain any cusp

A proof structure is ***correct*** if it has no switching cycle

= if every cycle has a cusp

Correctness





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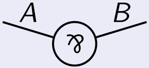

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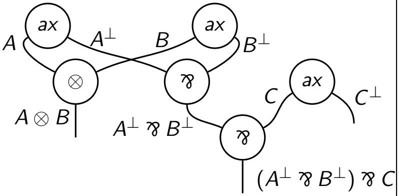

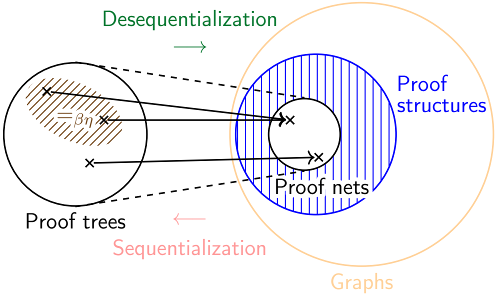

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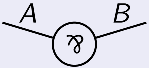

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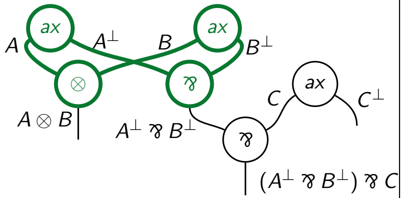

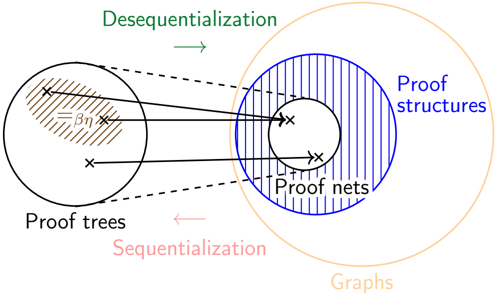

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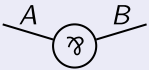

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Destination Sequentialization

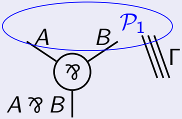
Sequentialization

Given a correct proof structure, there is a proof desequentializing to it.

How to prove it? One usual way: by finding a **splitting** vertex

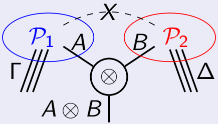
Splitting (terminal) $\otimes / \parr$ [Gir87]

- $\parr$ no vertex below
- $\otimes$ no vertex below & not in a cycle



$\rightsquigarrow$

$$\dfrac{\overset{\pi_1}{\vdash A, B, \Gamma}}{\vdash A \parr B, \Gamma}\ (\parr)$$



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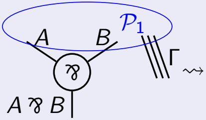

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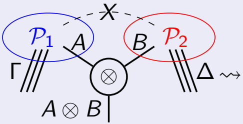
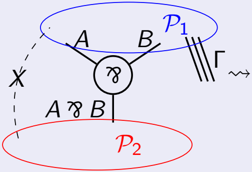


$$\dfrac{\overset{\pi_1}{\vdash A, \Gamma} \qquad \overset{\pi_2}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes)$$

Splitting $\parr$ (*aka* section) [DR89]

its conclusion edge is not in a cycle



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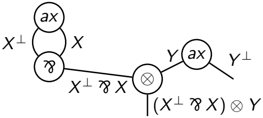
Sequentialization & Yeo's Theorem

Sequentialization

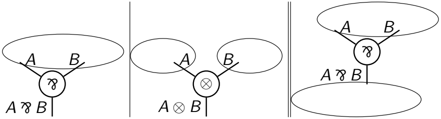
Proof nets



Cusp: a ⅋ and its two premises

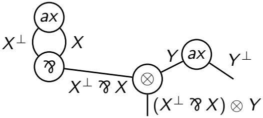
no switching / cusp-free cycle

$\Longrightarrow$ $\exists$ splitting vertex



Sequentialization

Proof nets

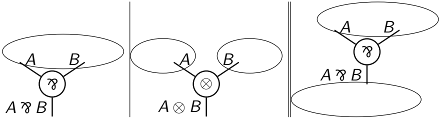


Cusp: a ⅋ and its two premises

no switching / cusp-free cycle

$\implies$ $\exists$ splitting vertex

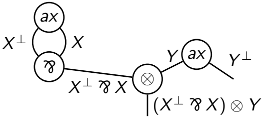
= is a cusp of all its cycles



Sequentialization & Yeo's Theorem

Sequentialization

Proof nets



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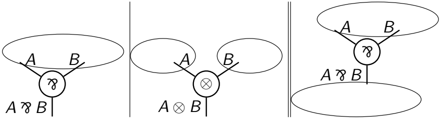
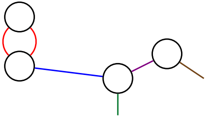


Yeo's Theorem

Edge-colored graphs

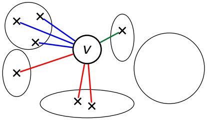


Cusp: a vertex and two of its edges of the same color

no cusp-free cycle

$\Longrightarrow$ $\exists$ splitting vertex

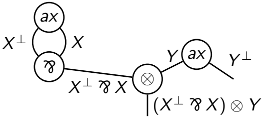
= is a cusp of all its cycles



Sequentialization & Yeo's Theorem

Sequentialization

Proof nets



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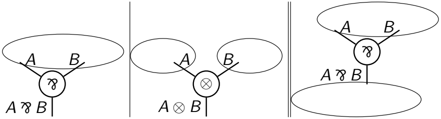
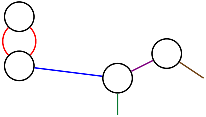


Yeo's Theorem

Edge-colored graphs



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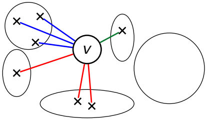


Encoding

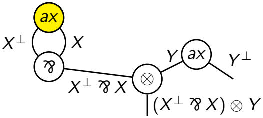
premises of a $\parr$ = same color

all other edges of different colors

Sequentialization & Yeo's Theorem

Sequentialization

Proof nets

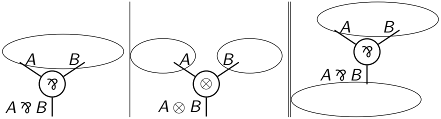


Cusp: a ⅋ and its two premises

no switching / cusp-free cycle

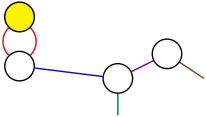
$\Longrightarrow$ $\exists$ splitting vertex

= is a cusp of all its cycles



Yeo's Theorem

Edge-colored graphs



Cusp: a vertex and two of its edges of the same color

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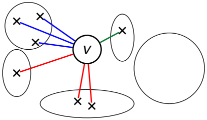


Encoding

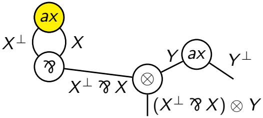
premises of a ⅋ = same color

all other edges of different colors

Sequentialization & Yeo's Theorem

Sequentialization

Proof nets



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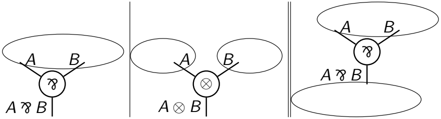
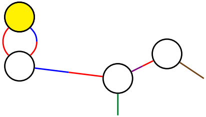


Generalized Yeo's Theorem

Half-Edge-colored graphs



Cusp: a vertex and two of its edges of the same color near it

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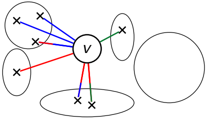

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Sequentialization & Yeo's Theorem

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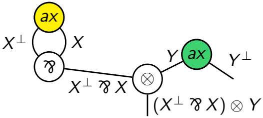
Proof nets



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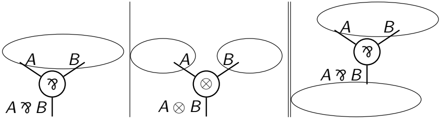

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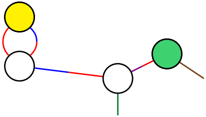

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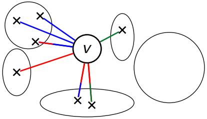

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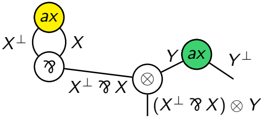

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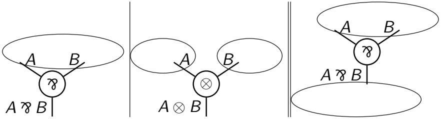

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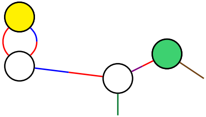

Cusp: a vertex and two of its edges of the same color near it

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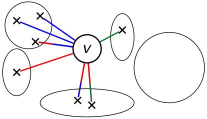
= is a cusp of all its cycles



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Outline

- **Multiplicative Linear Logic & Sequentialization**
  - Sequent Calculus & Proof Nets
  - Sequentialization from Yeo's theorem

- **Simple proof of (a generalized) Yeo's theorem**

Strict Partial Order on Vertices

Main idea: follow a path evidence of progression = a **strict partial order** $\lhd$

Goal: a $\lhd$-maximal vertex is splitting

Definition

$v \lhd u$ means there is a path p such that:

(1) p goes from v to u

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Strict Partial Order on Vertices × Colors

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$(v, \alpha) \lhd (u, \beta)$ means there is a path p such that:

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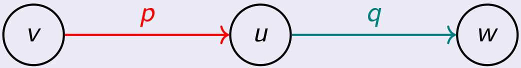
Proof: $\lhd$ is a strict partial order.

Irreflexivity: by definition.

Transitivity: if $(v, \alpha) \overset{p}{\lhd} (u, \beta) \overset{q}{\lhd} (w, \gamma)$ then $(v, \alpha) \overset{p \cdot q}{\lhd} (w, \gamma)$.

(1) ?

(2) ?



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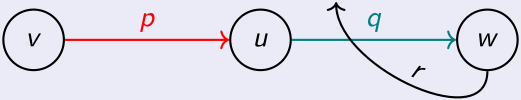

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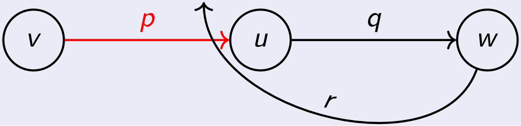


□

$\lhd$-maximal is splitting

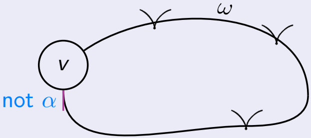
Lemma

In a graph with no cusp-free cycle, take v not splitting and α any color. Then $(v, \alpha) \lhd (u, \beta)$ for some (u, β) where β is the color of a cusp at u.

Proof.

v not splitting $\implies$ $\exists$ cycle ω with no cusp at v

- *w.l.o.g.* ω has a minimal number of cusps
- an edge of ω incident to v is not colored α



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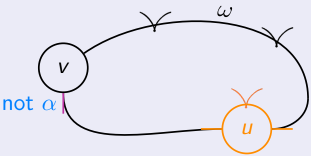

No cusp-free cycle:

set u the first cusp of ω, of color β

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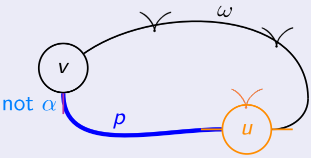

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$(v, \alpha) \overset{p}{\lhd} (u, \beta)$?

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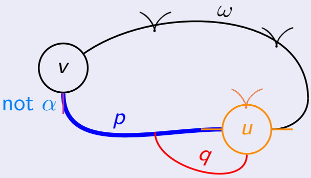
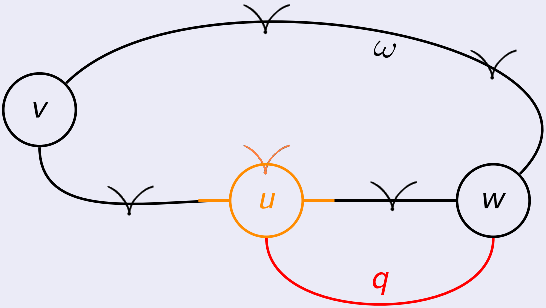


Key intermediate lemma

Cusp Minimization

Set ω a cycle with a cusp at u of color β but no cusp at v, and q a cusp-free path starting from u with color not β and ending on ω. Then either there is a cycle ω' with no cusp at v and strictly less cusps than ω or there exists a cusp-free cycle c.

Proof.



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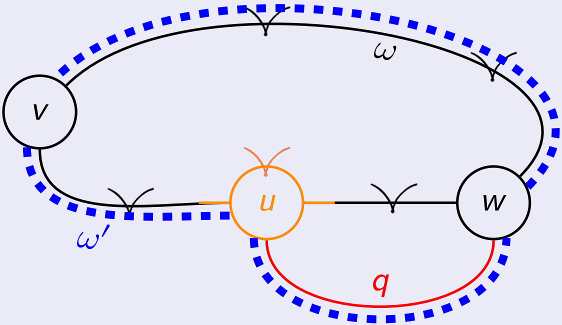

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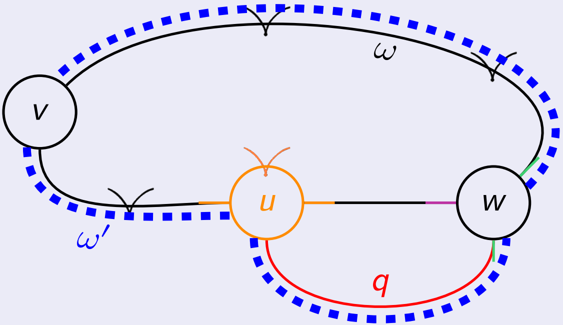

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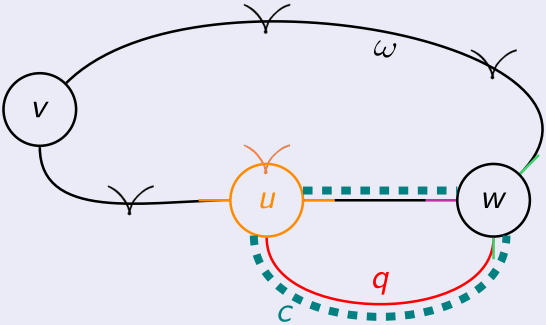

□

Generalized Yeo's Theorem

Generalized Yeo's Theorem

Take G a graph with an half-edge coloring and no cusp-free cycle.

Then G has a splitting vertex.

Proof.

A non-splitting vertex is smaller than some vertex in a cusp. □

Generalized Yeo's Theorem

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Take G a graph with an half-edge coloring and no cusp-free cycle.
Set P a subset of Vertices $\times$ Colors with

$$\{(v, \alpha) \mid \exists \text{ a cusp at } v \text{ of color } \alpha\} \subseteq P.$$

Then the vertex of any $\lhd$-maximal element of P is splitting.

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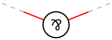
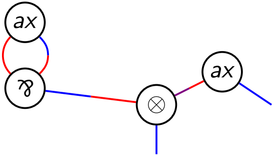
Back to proof nets: cusp = 



We get a:

Splitting $\parr$ with P all $\parr$-color pairs

Splitting terminal with $P := \{(v, \alpha) \mid v$ is a $\parr$ or $\otimes$ and α is the color of one of its premises$\}$

Splitting $\parr/\otimes$ with P all $\parr$- and $\otimes$-color pairs

Splitting $\parr/\otimes/ax$ with P all vertex-color pairs

Conclusion: Sequentialization and Graph Theory

Sequentialization [Gir87]

MLL *Proof nets are exactly the images of proof trees (with mix)*

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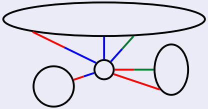

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Graph Theory

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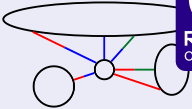

ROCQ CHECKED

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[Ngu20]

structural encoding

Kotzig [Kot59]

On perfect matchings

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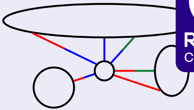

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On perfect matchings

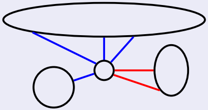
Grossman & Häggkvist [GH83]

Seymour & Giles [Sey78]

Shoesmith & Smiley [SS79]

Yeo [Yeo97]

A graph with no alternating cycle has a splitting vertex:



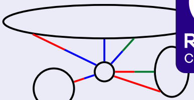
structural encoding

color encoding

color encoding

all by

Generalized Yeo



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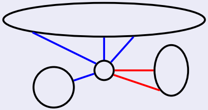

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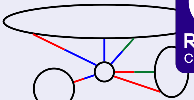


ROCQ CHECKED

(and a parameter)

Yeo with cycles

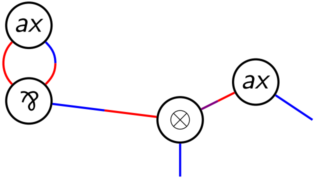
Allows some cusp-free cycles

color encoding

Proof Nets

Graph Theory

Thank you!



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[GH83] Jerrold W. Grossman and Roland Häggkvist. "Alternating Cycles in Edge-Partitioned Graphs". In: *Journal of Combinatorial Theory, Series B* 34.1 (1983), pp. 77–81. ISSN: 0095-8956. DOI: 10.1016/0095-8956(83)90008-4. URL: https://www.sciencedirect.com/science/article/pii/0095895683900084.

[Gir87] Jean-Yves Girard. "Linear logic". In: *Theoretical Computer Science* 50 (1987), pp. 1–102. DOI: 10.1016/0304-3975(87)90045-4.

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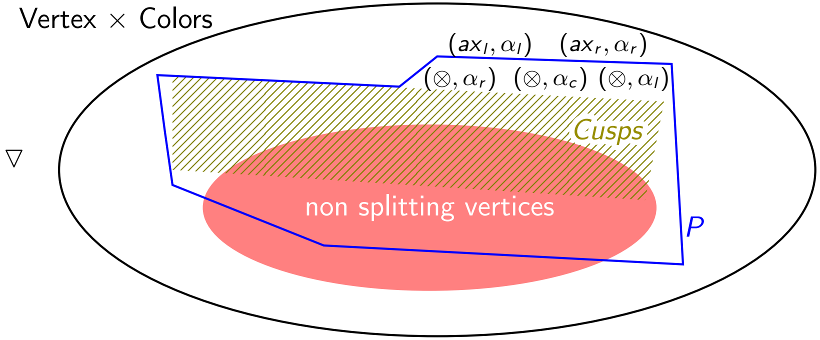
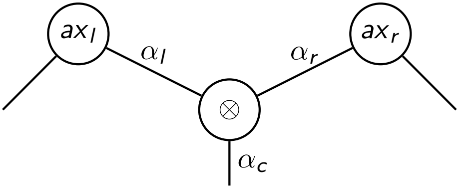
Interest of the parameter P





$$(\otimes, \alpha_l) \lhd (ax_r, \alpha_r)$$
$$(\otimes, \alpha_r) \lhd (ax_l, \alpha_l)$$
$$(\otimes, \alpha_c) \lhd (ax_r, \alpha_r)$$